Generalized Fractional Processes with Conditional Heteroscedasticity

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ABSTRACT

Generalized fractional processes in terms of Gegenbauer polynomials and GARCH (Generalized Autoregressive Conditional Heteroscedastic) errors is introduced and derived as a time series model. A related simulation study of the proposed model depicts statistical properties of the new class established in terms of the realization, sample autocorrelation function, theoretical autocorrelation function, partial autocorrelation function and the spectral density function

Key words: Long memory; Gegenbauer Polynomials; Heteroscedasticity; Fractional Difference; Volatility; Spectral Density; Stationarity; Invertibility.

1 Introduction

Existing literature suggests that the FARIMA model of Hosking [1], the generalized fractional Gegenbauer autoregressive moving-average (GARMA) model of Gray, Woodward and Zhang [2] and the autoregressive fractionally integrated moving average (ARFIMA) - generalized autoregressive conditional heteroscedastic (GARCH) model of Li and Ling [3] have unique time series characteristics of their own. But it is clear that in modelling of time series data, there is no systematic approach or a suitable, flexible class of time series models available in the literature to accommodate, analyze and forecast the changing frequency behaviour. Furthermore, in many financial time series modelling problems, it is known that heteroscedasticity plays an important role. In common use are the autoregressive conditional heteroscedastic (ARCH) models Engle [4] and their generalization, as well as the GARCH models due to Bollerslev [5]. Therefore in this paper consideration is given to a Gegenbauer process generated with GARCH(p,q) errors to model conditional heteroscedasticity.

It will be a fact that will be taken into consideration in presenting section 2 of this paper entailing the initial assumptions, notation, and preliminaries. Those initial conceptual paradigms will form the building blocks of time series model construction with respect to a new class. In view of this, a new class of time series models with a generalized fractional operator and conditional heteroscedasticity will be introduced in section 3 of this paper. Additionally, section 4 of the paper deals with the analysis of the new class with respect to the stationary case in terms of autocorrelation, partial autocorrelation and spectral density. Finally, in section 5 of this paper a conclusion is established using results obtained through the new class of time series models as a basis for further research.

2 Assumptions, Notation and Preliminaries

2.1 Fractional Differencing and Long Memory Models

Suppose that a long memory time series Y_t can be transformed to X_t through the filter

$$X_t = (I - B)^d Y_t,\ d \in (-0.5, 0.5)$$

to produce a short memory time series. The above differencing is known as fractional differencing. A class of models for long memory time series based on the above fractional differencing is given by

$$X_t = \sum_{i=1}^{p} \alpha_i X_{t-i} + Z_t - \sum_{j=1}^{q} \beta_j Z_{t-j}. \tag{2.1}$$

This can be written as

$$\alpha(B) X_t = \beta(B) Z_t.$$

where the polynomials $\alpha(B) = \sum_{i=1}^{p} \alpha_i B^i$ and $\beta(B) = \sum_{j=1}^{q} \beta_j B^j$ have zeros outside the unit circle and $\{Z_t\}$ is a sequence of uncorrelated random variables with mean zero and variance σ^2. The sequence $\{Z_t\}$ is known as the white noise and is denoted by $WN(0, \sigma^2)$.

It can be seen that

- the process $\{Y_t\}$ is stationary if $d < 1/2$,
- the process $\{Y_t\}$ is invertible if $d > -1/2$.
- the stationary process $\{Y_t\}$ is long memory if $0 < d < 1/2$.

2.2 Generalized Fractional Operator and its Properties

This section considers two types of generalized fractional operators.

Consider the process of the first type satisfying

$$(I - \alpha B)^{\delta} X_t = Z_t; \ -1 < \alpha < 1; \ \delta > 0\,. \tag{2.2}$$

This covers the standard AR(1) family when $\delta = 1$.

The model in (2.2) with GARCH errors had been considered by Peiris, Allen and Peiris [6] and Peiris and Thavaneswaran [7].

Now consider the next process type as a second order model which is defined:

$$(1 - \alpha_1 B - \alpha_2 B^2)^{\delta} X_t = Z_t, \tag{2.3}$$

where $1 - \alpha_1 z - \alpha_2 z^2 \neq 0$ for all $|z| \leq 1$, $\delta > 0$ and $\{Z_t\} \sim WN(0, \sigma^2)$.

We could write the model (2.3) as,

$$[(1 - \xi_1 B)(1 - \xi_2 B)]^{\delta} X_t = Z_t,$$

where $\xi_1 + \xi_2 = \alpha_1$ and $\xi_1 \xi_2 = -\alpha_2$. It can be shown that the solution of this model is given by,

$$X_t = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{\Gamma(k+\delta)\Gamma(l+\delta)\xi_1^k \xi_2^l}{\Gamma(k+1)\Gamma(l+1)\Gamma^2(\delta)} Z_{t-k-l},$$

where $\Gamma(\cdot)$ is the gamma function. The model in (2.3) is called the generalized second order autoregression with index δ. For simplicity, we call it GAR(2) model. It is clear that the class generated by (2.3) is more flexible than the standard AR(2) model. Although the autocorrelation function of the GAR(2) model can be obtained, it is not mathematically tractable as there is no closed form solution. Next section reports some interesting and mathematically elegant results of this class given by Shitan and Peiris [8].

For the model defined as in equation (2.3), it is easy to show that the spectral density $f(\omega)$ is given by,

$$f(\omega) = \frac{\sigma^2}{2\pi}[(1 + \alpha_1^2 + \alpha_2^2) - 2\alpha_1(1 - \alpha_2)\cos\omega - 2\alpha_2 \cos 2\omega]^{-\delta}.$$

2.3 Gegenbauer Polynomials and Properties

In mathematics, *Gegenbauer Polynomials* or *ultraspherical polynomials* $C_j^{\delta}(u),\ |u| \leq 1$ in terms of their generating functions are given as follows:

$$\frac{1}{(1-2ut+t^2)^{\delta}} = \sum_{j=0}^{\infty} C_j^{\delta}(u)t^j, \tag{2.4}$$

where

$$C_j^{\delta}(u) = \sum_{k=0}^{[j/2]}(-1)^k \frac{\Gamma(j-k+\delta)}{\Gamma(\delta)\Gamma(j+1)\Gamma(j-2k+1)}(2u)^{j-2k}$$

and $[j/2]$ stands for the integer part of $\frac{j}{2}$.

It is easy to verify that

- $C_0^{\delta}(u) = 1$
- $C_1^{\delta}(u) = 2\delta u$
- $C_2^{\delta}(u) = -\delta + 2\delta(1+\delta)u^2$
- $C_3^{\delta}(u) = -2\delta(1+\delta)u + \frac{4}{3}\delta(1+\delta)(2+\delta)u^3$

and $C_j^{\delta}(u)$ satisfies the recursion

$$C_j^{\delta}(u) = \frac{1}{j}[2u(j+\delta-1)C_{j-1}^{\delta}(u) - (j+2\delta-2)C_{j-2}^{\delta}(u)];\ j \geq 2.$$

However, the polynomial $1-2ut+t^2;\ |u| \leq 1$ has complex zeros which are on the unit circle. Therefore, we need certain restrictions on δ to ensure the square summability of $C_j^{\delta}(u)$ or $\sum_{j=0}^{\infty}\{C_j^{\delta}(u)\}^2 < \infty$.

2.4 Gegenbauer Process and its Properties

The process generated by (2.4)

$$(1-2uB+B^2)^{\delta}X_t = Z_t,$$

where $|u| \leq 1,\ \delta \neq 0$ and $\{Z_t\} \sim WN(0, \sigma^2)$

is called the *Gegenbauer Process* with parameters (u, δ) or Gegenbauer(u, δ).

It is clear that X_t has the following $MA(\infty)$ representation to the Gegenbauer Process given in (2.4):

$$X_t = \sum_{j=0}^{\infty} C_j^{\delta}(u) Z_{t-j}.$$

The next section considers a special case of the model (2.3).

In model (2.3), we need the assumption that the zeros of $1-\alpha_1 z-\alpha_2 z^2 = 0$ lie outside the unit circle or equivalently $\alpha_2 + \alpha_1 < 1$, $\alpha_2 - \alpha_1 < 1$ and $-1 < \alpha_2 < 1$ to ensure a stationary solution to (2.3). However, an interesting case arises when the above stationary assumption fails (or at least one of the above three inequalities does not hold).

2.5 The Gegenbauer Model

Now we investigate this special case of the process (2.3) given by

$$(1 - 2uB + B^2)^{\delta} X_t = Z_t, \tag{2.5}$$

where $|\alpha| \leq 1$, $\delta \neq 0$ and $\{Z_t\} \sim WN(0, \sigma^2)$.

The solution to this equation (2.4) can be obtained in terms of **Gegenbauer polynomials.**

2.6 Long Memory Gegenbauer Processes

It is easy to verify that the spectrum of the process given in (2.4) is

$$f_X(\omega) = |1 - 2uexp(i\omega) + exp(2i\omega)|^2 \sigma^2/\pi \tag{2.6}$$

or

$$f_X(\omega) = [4(cos\ \omega - u)]^{-2\delta} \sigma^2/\pi.$$

Clearly, the spectrum is unbounded at $\omega = arccos(u)$ when $\delta > 0$.

Therefore, we have the following results using the Theorem:

A stationary Gegenbauer Process has long memory

- for $|u| < 1$ if $0 < \delta < \frac{1}{2}$,
- for $|u| = 1$ if $0 < \delta < \frac{1}{4}$.

Now we state the following theorem for later reference for the existence of the above square summability of $C_j^\delta(u)$.

Theorem 1: Summability Theorem Gegenbauer polynomials defined by (2.5) above are square summable

- for $|u| < 1$ if $\delta < \frac{1}{2}$,
- for $|u| = 1$ if $\delta < \frac{1}{4}$.

Proof: The proof of the theorem can be established easily using the following lemma:

Lemma 1: Suppose that $(1 - \theta x)^{-\delta} = \sum_{j=0}^{\infty} \psi_j x^j$,

where

$$\psi_j = \theta^j \frac{\Gamma(j - \delta)}{\Gamma(j + 1)\Gamma(-\delta)}$$

and θ is a real or complex variable such that $|\theta| = 1$. Then

$$\sum_{j=0}^{\infty} |\psi_j^2| < \infty$$

if $\delta < \frac{1}{2}$.

Proof of the Theorem: Let

$$1 - 2ut + t^2 = (1 - \theta_1 t)(1 - \theta_2 t),$$

where $|u| < 1$ and $\theta_1, \theta_2 = u \pm i\sqrt{1 - u^2}; i = \sqrt{-1}$

such that $|t| = 1$.

Write $(1 - \theta_i t)^{-\delta} = \sum_{j=0}^{\infty} \psi_j^i t^j$; $i = 1, 2$. From the above lemma, it is clear that the coefficients satisfy

$$\sum_{j=0}^{\infty} |\psi_j^i|^2 < \infty$$

provided $\delta < \frac{1}{2}$.

When $|u| = 1$, we have $1 - 2t + t^2 = (1 - t)^2$. In this case

$$(1 - 2t + t^2)^{-\delta} = (1 - t)^{-2\delta} = \sum_{j=0}^{\infty} \psi_j^2 < \infty$$

if $2\delta < \frac{1}{2}$ or $\delta < \frac{1}{4}$.

3 A Generalized fractionally differenced model with Conditional heteroscedasticity

3.1 Gegenbauer Processes with GARCH Errors

In many financial time series modelling problems, it is known that heteroscedasticity plays an important role. Therefore, this section considers the Gegenbauer process generated by (2.3) $(1-2uB+B^2)^{\delta}X_t = Z_t$, with GARCH(p,q) errors to model the conditional heteroscedasticity. That is, $Z_t|F_{t-1} \sim N(0, h_t^2)$, where F_{t-1} is the history of the process and h_t^2 satisfies the GARCH(p,q) process given by

$$h_t^2 = \alpha_0 + \sum_{j=1}^{p} \alpha_j Z_{t-j}^2 + \sum_{j=1}^{q} \beta_j h_{t-j}^2 \tag{3.1}$$

The original process generated by (2.3) and (2.6) could now be introduced with GARCH(p,q) errors as a *Gegenbauer*(u,δ)-*GARCH*(p,q) process created as a result of generalized fractional differencing coupled with conditional heteroskedasticity.

This original *Gegenbauer*(u,δ)-*GARCH*(p,q) time series model governed by

$$(1-2uB+B^2)^{\delta}X_t = Z_t, \tag{3.2}$$

and it is easy to verify that the spectrum of the process given in (3.8) is

$$f_X(\omega) = |1 - 2uexp(i\omega) + exp(2i\omega)|^2\sigma^2/\pi \tag{3.3}$$

or

$$f_X(\omega) = [4(cos\ \omega - u)]^{-2\delta}\sigma^2/\pi.$$

Clearly, the spectrum is unbounded at $\omega = arccos(u)$ when $\delta > 0$.

Furthermore, the theoretical (true) autocorrelation function (acf) of the above model with a stationary filter could be found as a closed form solution through the expression

$$[E(X_k X_{k+t})] = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} C_j C_i [E(Z_{t-j})(Z_{k+t-i})]$$

Which could then be simplified in terms of Gegenbauer coefficients to the expression

$$[E(X_k X_{k+t})] = \sum_{j=0}^{\infty} C_j C_{j+t} \quad (3.4)$$

The conceptual results of (3.2),(3.3) and (3.4) will be the topic of interest in terms of simulation results for differing sets of statistical parameters in the next section of this paper.

4 Analytical simulation results of new class

Simulation results depicting realizations of length 400 (top left plot), sample autocorrelation (top right plot), true autocorrelation (middle left plot), partial autocorrelation (middle right plot) and spectral density (bottom plot) for parameters are shown below as figures 1 to 4.

The simulation results will provide a visual display of properties such as volatility, long memory and conditional heteroskedasticity.

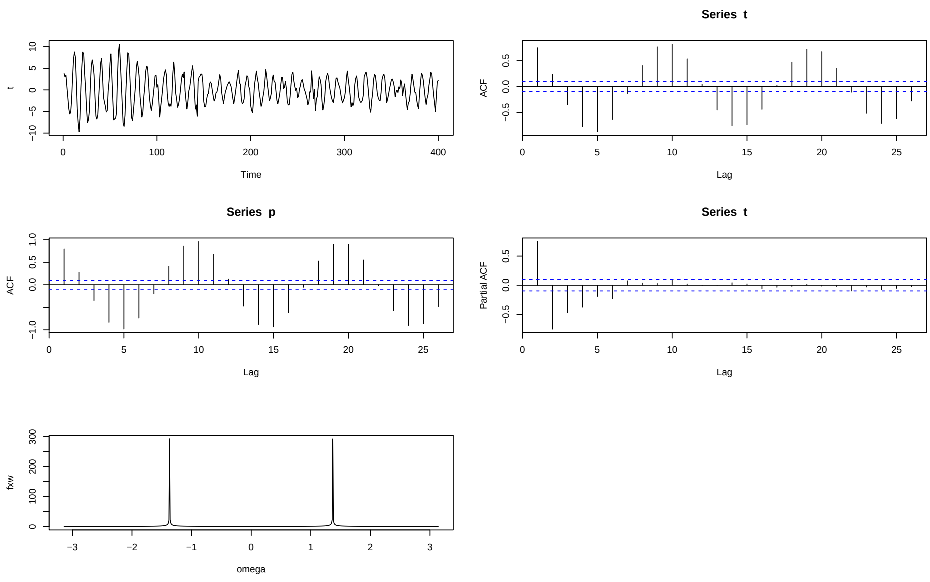

Figure 1: $u = 0.8$, $\delta = 0.45$, $\alpha_0 = 0.4$, $\alpha_1 = 0.3$, $\beta = 0.3$

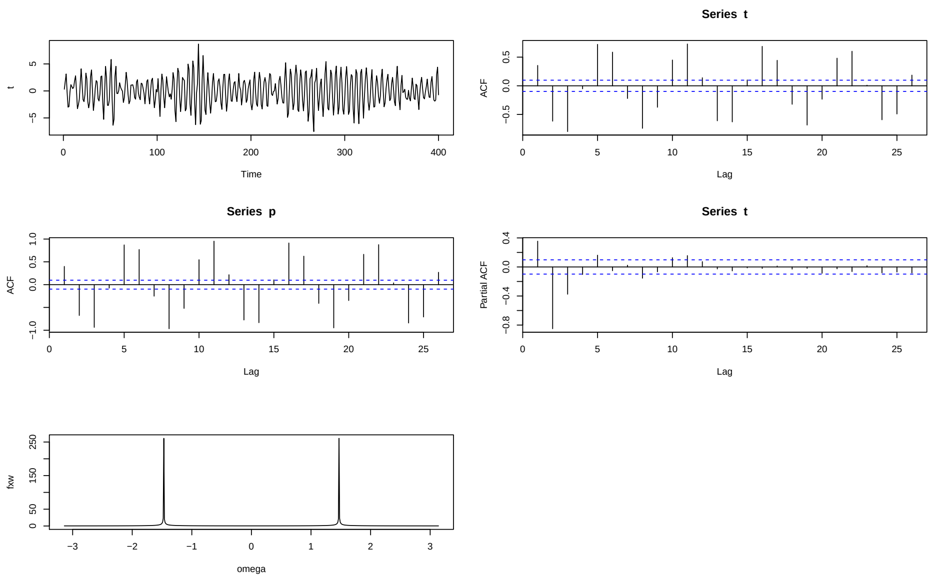

Figure 2: $u = 0.4$, $\delta = 0.45$, $\alpha_0 = 0.4$, $\alpha_1 = 0.3$, $\beta = 0.3$

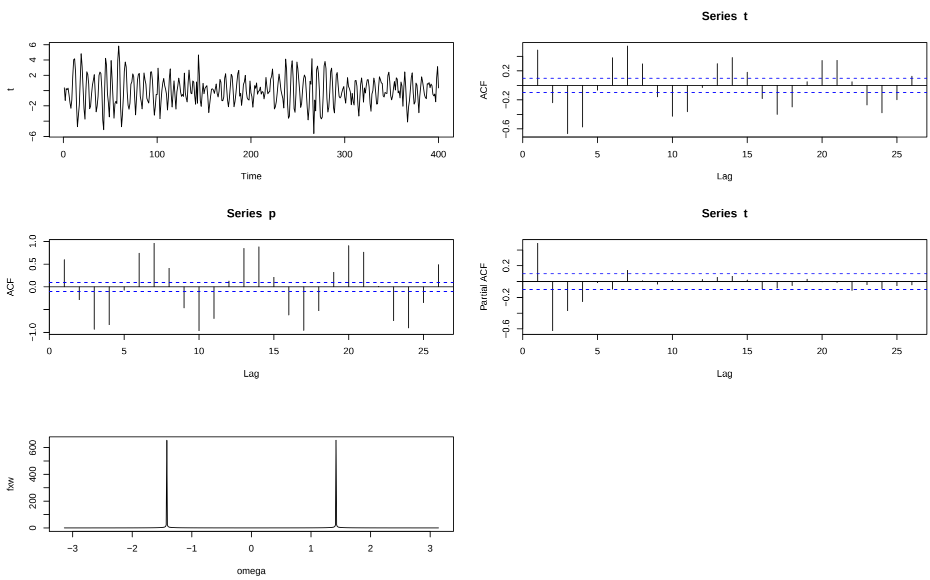

Figure 3: $u = 0.6$, $\delta = 0.45$, $\alpha_0 = 0.4$, $\alpha_1 = 0.3$, $\beta = 0.3$

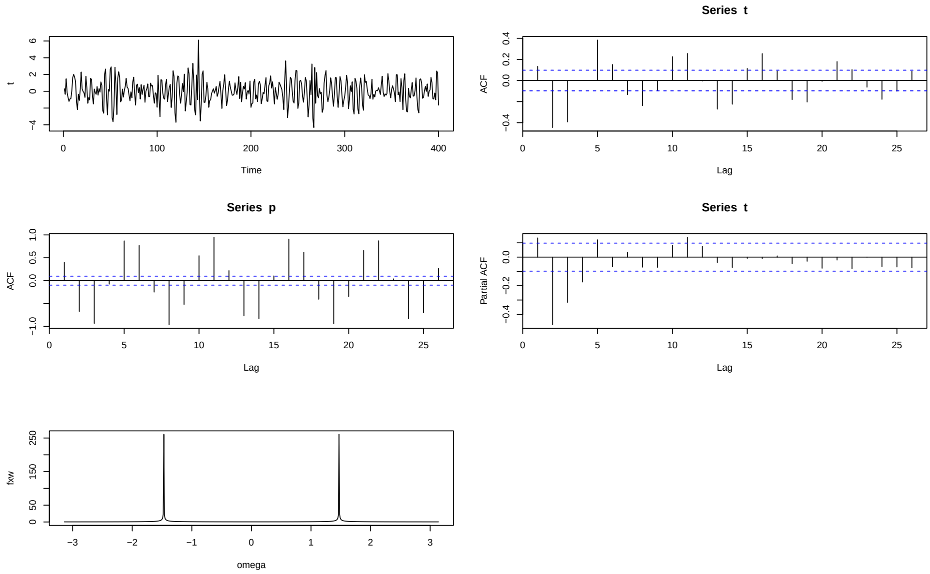

Figure 4: $u = 0.4$, $\delta = 0.10$, $\alpha_0 = 0.4$, $\alpha_1 = 0.3$, $\beta = 0.3$

In all four figures above, it is evident that the realization plots show a very high degree of volatility and conditional heteroskedasticity. By inspecting the sample and theoretical autocorrelation it is clear that they are approximately the same implying the simulations are reasonably accurate. Furthermore, it is clear that all the partial autocorrelation functions above decay slowly at a hyperbolic rate characterizing long memory. It is corroborated by the spectral density function with an unbounded high peak at the frequency origin symbolizing long memory.

5 Conclusion

The results of this paper establishes the existence of a creative component in terms of a Generalized Fractionally differenced time series model with conditional heteroskedasticity. Fractional differencing with the Gegenbauer polynomials and the incorporation of GARCH errors introduces an irregular random noise creating a high degree of volatility as well as long memory based on past information, and provides a base for future research.

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