Statistical Analysis for a Three Service Point Tandem Queue with Blocking

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ABSTRACT

A maximum likelihood estimator (MLE), a consistent asymptotically normal (CAN) estimator and asymptotic confidence limits for the expected number of entities in the system in a three service point tandem queue with blocking and busy service point 1 and zero queue capacity in front of service points 2 and 3 are obtained.

Keywords: CAN estimator; Expected number of entities in the system; Multivariate central limit theorem; Slutsky theorem-tandem queue.

1. Introduction

Most of the studies on several queueing models are confined to only obtaining expressions for transient or stationary (steady state) solutions and do not consider the associated statistical inference problems. Nowadays, statistical analysis of queueing systems is an important area of research in Queueing theory. Parametric estimation, Interval estimation and Bayesian estimation are some of the essential tools to understand any random phenomena using stochastic models. Analysis of queueing systems in this direction has not received much attention in the past. Whenever the systems are fully observable in terms of their random components such as interarrival times and service times, standard parametric techniques of statistical theory are quite appropriate. Recently, Narayan Bhat (2003) has provided an overview of methods available for estimation, when the information is restricted to the number of entities in the system at some discrete points in time. Narayan Bhat (2003) has also described how maximum likelihood estimation is applied directly to the underlying Markov chain in the queue length process in M/G/1 and GI/M/1 queues. Table 1 indicates the present state of work of queueing systems, wherein the asymptotic confidence limits for measures of system performance are obtained.

Generally speaking, the queueing models assume that each service channel consists of only one service point. Situations do exist, where each service channel may consist of several service points in series. In this situation, an entity must pass through all these service points in succession before completing its service. Such situations are known as queues in series or tandem queues. e.g., (a) in a manufacturing process, units must pass through a series of service points (work stations), where each service point performs a given task or job, (b) in a university registration process, each registrant must pass through a series of counters such as advisor, department chairman and cashier, (c) in a clinical physical examination procedure, a patient must go through a series of stages such as laboratory tests, ECG and chest X-ray. In all these models, it is not only sufficient to know how many entities are there in the system but also where they are.

Table 1: Present state of work for queueing systems

S. No.	**System Description**	**Authors**	**Confidence limits obtained for**
1	M/M/1	Clarke (1957)	MLEs of λ and μ.
2	$M/M/1/\infty$ and M/M/1/N	Yadavalli et al (2004)	W_Q
3	$M/M/c/\infty$ and M/M/c/N	Yadavalli et al (2006)	W_Q
4	Tandem queue with blocking and dependent structure for service times	Chandrasekhar et al (2006)	L_s, W_s
5	Tandem queue with blocking	Chandrasekhar et al (2008)	W_s
6	Bulk-arrival queueing system $M^{[x]}/M/1$	Paul Savariappan and Chandrasekhar (2009a)	L_s
7	Tandem and bulk service queueing systems	Paul Savariappan et al (2009 b)	L_s
8	M/M/R queue with heterogeneous servers	Tsung-Yin Wang et al (2006)	L_s
9	Two station and three station tandem queues with blocking	Chandrasekhar and Vaidyanathan (2009)	L_s

In general, in all the above mentioned tandem queueing models studied so far, it is assumed that no queues are allowed in front of service points. This is not so in any real life situation. Hence, an attempt is made in this paper to study in detail a three service point tandem queue, where service point 1 is busy but no queue is allowed at service point 2 and service point 3. A MLE, CAN and asymptotic confidence limits for the expected number of entities in the system are also obtained.

2. System Description and Assumptions

Consider a simplified one channel queueing system consisting of three series service points as below :

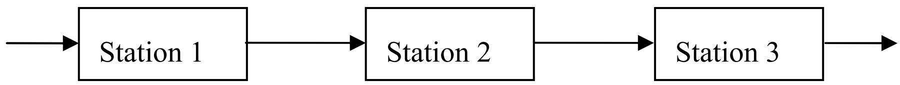

Figure 1: System configuration.

An entity arriving for service must pass through service point 1, service point 2 and service point 3 before completing its service. The assumptions of the model are as follows:

i. Service times at service point 1, service point 2 and service point 3 are independent and exponentially distributed with service rates μ_1, μ_2 and μ_3 respectively.

ii. Service point 1 is busy but no queue is allowed at service point 2 and service point 3.

iii. Service point 2 and service point 3 are either free or busy.

iv. Service point i (i=1, 2) is said to be blocked, if the entity in service point i(i=1, 2) completes its service before service point (i+1) (i=1, 2) becomes free. In this case, the entity cannot wait between the service points i and (i+1) (i=1, 2), since this is not allowed and the entity remains in service point i(i=1, 2) itself.

3. Analysis of the System

Let the symbols 0, 1, and b represent free, busy and blocked states of a service point. Let X(t), Y(t) and Z(t) respectively denote the states of service point 1, service point 2 and service point 3 and the vector process $W(t)=\{(X(t),Y(t),Z(t)),t\geq 0\}$ with state space

$$E = \{(1,0,0),(1,1,0),(1,0,1),(1,1,1),(b,1,0),(1,b,1),(b,1,1),(b,b,1)\} \tag{3.1}$$

the state of the system at time t. Since the interarrival and service times are all exponential, it clearly follows that the process W(t) is a Markov process with the infinitesimal generator given by

$$Q = \begin{array}{c} \\ (1,0,0) \\ (1,1,0) \\ (1,0,1) \\ (1,1,1) \\ (b,1,0) \\ (1,b,1) \\ (b,1,1) \\ (b,b,1) \end{array} \begin{array}{cccccccc} (1,0,0) & (1,1,0) & (1,0,1) & (1,1,1) & (b,1,0) & (1,b,1) & (b,1,1) & (b,b,1) \\ -\mu_1 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -(\mu_1+\mu_2) & \mu_2 & 0 & \mu_1 & 0 & 0 & 0 \\ \mu_3 & 0 & -(\mu_1+\mu_3) & \mu_1 & 0 & 0 & 0 & 0 \\ 0 & \mu_3 & 0 & -(\mu_1+\mu_2+\mu_3) & 0 & \mu_2 & \mu_1 & 0 \\ 0 & 0 & 0 & \mu_2 & -\mu_2 & 0 & 0 & 0 \\ 0 & 0 & \mu_3 & 0 & 0 & -(\mu_1+\mu_3) & 0 & \mu_1 \\ 0 & 0 & 0 & 0 & \mu_3 & 0 & -(\mu_2+\mu_3) & \mu_2 \\ 0 & 0 & 0 & \mu_3 & 0 & 0 & 0 & -\mu_3 \end{array} \tag{3.2}$$

Let
$a=(1,0,0), b'=(1,1,0), c=(1,0,1), d=(1,1,1), e=(b,1,0), f=(1,b,1), g=(b,1,1)$ *and* $h=(b,b,1)$.
Also, let $p_{ijk}(t) = \Pr[W(t) = (i,j,k)]\ \forall (i,j,k) \in E$ represent the probability that the system is in state (i,j,k) at time t with the initial condition $p_{100}(0)=1$. From the infinitesimal generator given in (3.2), we have the following system of differential - difference equations:

$$\frac{dp_a(t)}{dt} = -\mu_1 p_a(t) + \mu_3 p_c(t) \tag{3.3}$$

$$\frac{dp_{b'}(t)}{dt} = \mu_1 p_a(t) - (\mu_1+\mu_2) p_b(t) + \mu_3 p_d(t) \tag{3.4}$$

$$\frac{dp_c(t)}{dt} = \mu_2 p_b(t) - (\mu_1+\mu_3) p_c(t) + \mu_3 p_f(t) \tag{3.5}$$

$$\frac{dp_d(t)}{dt} = \mu_1 p_c(t) - (\mu_1+\mu_2+\mu_3) p_d(t) + \mu_2 p_e(t) + \mu_3 p_h(t) \tag{3.6}$$

$$\frac{dp_e(t)}{dt} = \mu_1 p_b(t) - \mu_2 p_e(t) + \mu_3 p_g(t) \tag{3.7}$$

$$\frac{dp_f(t)}{dt} = \mu_2 p_d(t) - (\mu_1+\mu_3) p_f(t) \tag{3.8}$$

$$\frac{dp_g(t)}{dt} = \mu_1 p_d(t) - (\mu_2 + \mu_3) p_g(t) \tag{3.9}$$

$$\frac{dp_h(t)}{dt} = \mu_1 p_f(t) + \mu_2 p_g(t) - \mu_3 p_h(t) \tag{3.10}$$

3.1 Steady state solution

The equations (3.3) - (3.10) can be solved using the fact that $\sum_{(i,j,k)\in E} p_{ijk}(t) = 1$.

Since we wish to study the stationary behaviour of the system, let $\lim_{t\to\infty} p_{ijk}(t) = p_{ijk}$.

Let $\underline{p} = (p_a, p_{b'}, p_c, p_d, p_e, p_f, p_g, p_h)$ be the stationary distribution corresponding to the Markov process $\{W(t), t \ge 0\}$. The steady state probabilities are given by

$$p_a = \frac{\mu_2^2\mu_3^3(\mu_1+\mu_2)(\mu_2+\mu_3)(2\mu_1+\mu_2+\mu_3)}{\{(\mu_1+\mu_2+\mu_3)f(\mu_1,\mu_2,\mu_3)+g(\mu_1,\mu_2,\mu_3)\}} \tag{3.11}$$

$$p_{b'} = \frac{\mu_1\mu_2\mu_3^2(\mu_2+\mu_3)\left[(\mu_1+\mu_2+\mu_3)\left(\mu_1^2+\mu_2\mu_3+\mu_3\mu_1\right)+\mu_1\mu_2\mu_3\right]}{\{(\mu_1+\mu_2+\mu_3)f(\mu_1,\mu_2,\mu_3)+g(\mu_1,\mu_2,\mu_3)\}} \tag{3.12}$$

$$p_c = \frac{\mu_1\mu_2^2\mu_3^2(2\mu_1+\mu_2+\mu_3)(\mu_1+\mu_2)(\mu_2+\mu_3)}{\{(\mu_1+\mu_2+\mu_3)f(\mu_1,\mu_2,\mu_3)+g(\mu_1,\mu_2,\mu_3)\}} \tag{3.13}$$

$$p_d = \frac{\mu_1^2\mu_2\mu_3(\mu_1+\mu_2)(\mu_2+\mu_3)(\mu_3+\mu_1)(\mu_1+\mu_2+\mu_3)}{\{(\mu_1+\mu_2+\mu_3)f(\mu_1,\mu_2,\mu_3)+g(\mu_1,\mu_2,\mu_3)\}} \tag{3.14}$$

$$p_e = \frac{\mu_1^2\mu_3^2[\mu_2\mu_3(\mu_2+\mu_3)(2\mu_1+\mu_2+\mu_3)+\mu_1(\mu_1+\mu_3)(\mu_1+\mu_2+\mu_3)(\mu_1+2\mu_2+\mu_3)]}{\{(\mu_1+\mu_2+\mu_3)f(\mu_1,\mu_2,\mu_3)+g(\mu_1,\mu_2,\mu_3)\}} \tag{3.15}$$

$$p_f = \frac{\mu_1^2\mu_2^2\mu_3(\mu_1+\mu_2)(\mu_2+\mu_3)(\mu_1+\mu_2+\mu_3)}{\{(\mu_1+\mu_2+\mu_3)f(\mu_1,\mu_2,\mu_3)+g(\mu_1,\mu_2,\mu_3)\}} \tag{3.16}$$

$$p_g = \frac{\mu_1^3\mu_2\mu_3(\mu_1+\mu_2)(\mu_3+\mu_1)(\mu_1+\mu_2+\mu_3)}{\{(\mu_1+\mu_2+\mu_3)f(\mu_1,\mu_2,\mu_3)+g(\mu_1,\mu_2,\mu_3)\}} \tag{3.17}$$

$$p_h = \frac{\mu_1^3\mu_2^2(\mu_1+\mu_2)(\mu_1+\mu_2+\mu_3)(\mu_1+\mu_2+2\mu_3)}{\{(\mu_1+\mu_2+\mu_3)f(\mu_1,\mu_2,\mu_3)+g(\mu_1,\mu_2,\mu_3)\}} \tag{3.18}$$

where

$$\begin{aligned} f(\mu_1,\mu_2,\mu_3) &= (\mu_1+\mu_2)\{\ \mu_2^2\mu_3(\mu_2+\mu_3)(\mu_1^2+\mu_1\mu_3+\mu_3^2) \\ &+ \mu_1^2\mu_2\mu_3(\mu_3+\mu_1)(\mu_1+\mu_2+\mu_3)+\ \mu_1^3\mu_2^2(\mu_1+\mu_2+2\mu_3)\ \}+ \\ &\mu_1\mu_2\mu_3^2(\mu_2+\mu_3)(\mu_1^2+2\mu_1\mu_3+\mu_2\mu_3)+\ \mu_1^3\mu_3^2(\mu_3+\mu_1)(\mu_1+2\mu_2+\mu_3) \end{aligned} \tag{3.19}$$

and

$$g(\mu_1,\mu_2,\mu_3) = \mu_1\mu_2\mu_3^2(\mu_1+\mu_2)(\mu_2+\mu_3)[\mu_1\mu_2+\mu_2\mu_3+\mu_3\mu_1] \tag{3.20}$$

3.2. Expected number of entities in the system

The expected number of entities in the system is given by

$$L_s = \sum_{n=o}^{\infty} n.p_n = 1(p_a)+2(p_{b'}+p_c+p_e)+3(p_d+p_f+p_g+p_h)$$

Substituting for $p_a, p_{b'}, p_c,\ p_d,\ p_e,\ p_f,\ p_g, p_h$, it can be shown that

$$L_s = \frac{\{(\mu_1+\mu_2+\mu_3)f_1(\mu_1,\mu_2,\mu_3)+g_1(\mu_1,\mu_2,\mu_3)\}}{\{(\mu_1+\mu_2+\mu_3)f(\mu_1,\mu_2,\mu_3)+g(\mu_1,\mu_2,\mu_3)\}} \tag{3.21}$$

where

$$\begin{aligned} f_1(\mu_1,\mu_2,\mu_3) &= \{\ (\mu_2+\mu_3)[\mu_2^2\mu_3^3(\mu_1+\mu_2)+2\mu_1^2\mu_2\mu_3^2(\mu_3+\mu_1)+2\mu_1^2\mu_2\mu_3^3]+2\mu_1^3\mu_3{}^2\mu_2(\mu_3+\mu_1)\ \} \\ &+(\mu_1+\mu_2+\mu_3)[3\mu_1^3\mu_2^2(\mu_1+\mu_2)+2\mu_1\mu_2^2\mu_3^2(\mu_2+\mu_3)+2\mu_3^2\mu_1^3(\mu_3+\mu_1)] \\ &+\ 3\mu_1^2\mu_2\mu_3(\mu_1+\mu_2)(\mu_1+\mu_2+\mu_3)^2 \end{aligned}$$

and

$$g_1(\mu_1,\mu_2,\mu_3) = \mu_1\mu_2\mu_3^2(\mu_2+\mu_3)\{\mu_2(\mu_1+\mu_2)(2\mu_1+\mu_3)+2\mu_1\mu_3(\mu_1+\mu_2)\}.$$

Further, $f(\mu_1,\mu_2,\mu_3)$ and $g(\mu_1,\mu_2,\mu_3)$ are given in (3.19) and (3.20).

Special Case

By choosing $\mu_1 = \mu_2 = \mu_3 = \mu$ (say), i.e., the service times at service point 1, service point 2 and service point 3 are independent and identically distributed (i.i.d) exponential variates with the parameter μ, it is readily seen that

$$p_a = \frac{4}{39}, p_{b'} = \frac{5}{39}, p_c = \frac{4}{39},\ p_d = \frac{6}{39},\ p_e = \frac{8}{39},\ p_f = \frac{3}{39},\ p_g = \frac{3}{39},$$
$$p_h = \frac{6}{39} \text{ and } L_s = \frac{92}{39} \qquad (3.22)$$

In the next section, we obtain the MLE and CAN for the expected number of entities in the system.

3.3 MLE and CAN Estimator for the Expected Number of Entities in the System

3.3.1. ML Estimator

Let $Y_{11}, Y_{12}, ..., Y_{1n};\ Y_{21}, Y_{22}, ..., Y_{2n};\ Y_{31}, Y_{32}, ..., Y_{3n}$ be three random samples of size n, each drawn from three different exponential service time populations with the parameters μ_1, μ_2 and μ_3 respectively. It is clear that

$$E\left(\overline{Y}_i\right) = \frac{1}{\mu_i},\ \ i = 1,2,3 \text{ where } \overline{Y}_i = \sum_{j=1}^{n} Y_{ij} / n,\ \ i = 1,2,3 \text{ and } \overline{Y}_1, \overline{Y}_2, \text{ and } \overline{Y}_3$$

are the sample means of service times at service point 1, service point 2 and service point 3 respectively. It can be shown that $\overline{Y}_i$, $i = 1,2,3$ is the MLE of $\frac{1}{\mu_i}$, $i = 1,2,3$. Let $\theta_1 = \frac{1}{\mu_1}$, $\theta_2 = \frac{1}{\mu_2}$ and $\theta_3 = \frac{1}{\mu_3}$. Clearly, the expected number of entities in the system given in (3.21) yields

$$L_s = \ell_1 / \ell_2 \qquad (3.23)$$

where

$$\ell_1 = \theta_1^3\theta_2\theta_3(\theta_1+\theta_2)(\theta_2+\theta_3)(\theta_1+2\theta_2+2\theta_3) + \theta_1\{(\theta_2+\theta_3)[\theta_1^3(\theta_1+\theta_2) + 2\theta_1\theta_2^2(\theta_1+\theta_3) + 2\theta_1^2\theta_2^2] + 2\theta_2^3\theta_3(\theta_1+\theta_3)\}$$
$$(\theta_1\theta_2+\theta_2\theta_3+\theta_3\theta_1) + (\theta_1\theta_2+\theta_2\theta_3+\theta_3\theta_1)^2[2\theta_1^3(\theta_2+\theta_3) + 2\theta_2^3(\theta_3+\theta_1) + 3\theta_3^3(\theta_1+\theta_2)]$$
$$+ 3\theta_3(\theta_1+\theta_2)(\theta_1\theta_2+\theta_2\theta_3+\theta_3\theta_1)^3$$

and

$$\ell_2 = (\theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_1)\{ (\theta_1 + \theta_2)\left[\theta_1^2(\theta_2 + \theta_3)\left(\theta_1^2 + \theta_1\theta_3 + \theta_3^2\right) + \theta_2\theta_3(\theta_1 + \theta_3)(\theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_1) + \theta_3^3(2\theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_1)\right]$$
$$+ \theta_1^2\theta_2(\theta_2 + \theta_3)\left(2\theta_1\theta_2 + \theta_2\theta_3 + \theta_1^2\right) + \theta_2^3(\theta_1 + \theta_3)(\theta_1\theta_2 + \theta_2\theta_3 + 2\theta_3\theta_1) \} + \theta_1^3\theta_2\theta_3(\theta_1 + \theta_2)(\theta_2 + \theta_3)(\theta_1 + \theta_2 + \theta_3)$$

and hence MLE of L_s is obtained by replacing θ_1, θ_2 and θ_3 by the corresponding estimators namely $\overline{Y}_1, \overline{Y}_2$ and $\overline{Y}_3$ respectively. It may be noted that $\hat{L}_s$ is a real valued function in $\overline{Y}_i$, $(i = 1,2,3)$, which is also differentiable.

Special cases:

(i) Suppose $\theta_1 = \theta_2 = \theta_3 = \theta$ (say).i.e., the service times at service point 1, service point 2 and service point 3 are i.i.d exponential variates each with the parameter θ, it is seen from (3.23) that $L_s = \frac{92}{39}$, which is in agreement with (3.22).

(ii) Suppose $\theta_i = i\,\theta$, i=1,2,3. From (3.23),it is further seen that

$$L_s = \frac{79761}{29118} = 2.7392.$$

(iii) Finally, if θ_i =(3-i) θ, i= 1,2,3, the expected number of customers in the system reduces from (3.23) to $L_s = \frac{10}{7}$.

3.3.2 A CAN estimator using the multivariate central limit theorem

Suppose $T_1', T_2', T_3', \ldots$ are independent and identically distributed k dimensional random variables such that

$$T_n' = (T_{1n}, T_{2n}, T_{3n}, \ldots, T_{kn}), \quad n = 1,2,3,\ldots$$

having the first and second order moments $E(T_n) = \mu$ and $\mathrm{var}(T_n) = \sum$.

Define the sequence of random variables

$$\overline{T}_n' = \left(\overline{T}_{1n}, \overline{T}_{2n}, \overline{T}_{3n}, \ldots, \overline{T}_{kn}\right), \quad n = 1,2,3,\ldots$$

$$\text{where } \overline{T}_{in} = \frac{\sum_{j=1}^{n} T_{ij}}{n}, \quad i = 1,2,3,...,k.$$

Then, $\sqrt{n}\left(\overline{T}_n - \mu\right) \xrightarrow{d} N_k\left(0, \sum\right)$ as $n \to \infty$.

By applying the multivariate central limit theorem given above, it is readily seen that

$$\sqrt{n}\left[\left(\overline{Y}_1, \overline{Y}_2, \overline{Y}_3\right) - \left(\theta_1, \theta_2, \theta_3\right)\right] \xrightarrow{d} N_3\left(0, \sum\right) \text{ as } n \to \infty,$$

where the dispersion matrix $\sum = \left(\left(\sigma_{ij}\right)\right)$ is given by $\sum = \text{diag}\left(\theta_1^2, \theta_2^2, \theta_3^2\right)$.

Again from Radhakrishna Rao (1974), we have

$$\sqrt{n}\left(\hat{L}_s - L_s\right) \xrightarrow{d} N\left(0, \sigma^2(\theta)\right) \text{ as } n \to \infty, \text{ where } \theta = \left(\theta_1, \theta_2, \theta_3\right) \text{ and}$$

$$\sigma^2(\theta) = \sum_{i=1}^{3}\left(\frac{\partial L_s}{\partial \theta_i}\right)^2 \sigma_{ii} = \sum_{i=1}^{3}\left(\frac{\partial L_s}{\partial \theta_i}\right)^2 \theta_i^{\,2} \tag{3.24}$$

By substituting the partial derivatives $\left(\frac{\partial L_s}{\partial \theta_i}\right)$, i=1, 2, 3 in (3.24), we obtain an expression for $\sigma^2(\theta)$. Thus, $\hat{L}_S$ is a CAN estimator of L_S. There are several methods for generating CAN estimators and the Method of moments and the Method of Maximum likelihood are commonly used to generate such estimators. (see Sinha(1986)).

3.4 Confidence Limits for the Expected Number of Entities in the System

Let $\sigma^2(\hat{\theta})$ be the estimator of $\sigma^2(\theta)$ obtained by replacing θ by a consistent estimator $\hat{\theta}$ namely $\hat{\theta} = \left(\overline{Y}_1, \overline{Y}_2, \overline{Y}_3\right)$. Let $\hat{\sigma}^2 = \sigma^2(\hat{\theta})$. Since $\sigma^2(\theta)$ is a continuous function of θ, $\hat{\sigma}^2$ is a consistent estimator of $\sigma^2(\theta)$. i.e., $\hat{\sigma}^2 \xrightarrow{P} \sigma^2(\theta)$ as $n \to \infty$.

By Slutsky theorem $\left(X_n \xrightarrow{d} X,\;\; Y_n \xrightarrow{P} b \Rightarrow \frac{X_n}{Y_n} \xrightarrow{d} \frac{X}{b},\; b \neq 0 \right)$,

we have

$$\sqrt{n}\left(\frac{\hat{L}_S - L_S}{\hat{\sigma}} \right) \xrightarrow{d} N(0,1). \quad \text{i.e.,} \quad \Pr\left[-k_{\alpha/2} < \frac{\sqrt{n}\left(\hat{L}_S - L_S \right)}{\hat{\sigma}} < k_{\alpha/2} \right] = (1-\alpha),$$

where $k_{\alpha/2}$ is obtained from normal tables. Hence, a $100(1-\alpha)\%$ asymptotic confidence interval for L_S is given by

$$\hat{L}_S \pm k_{\alpha/2} \frac{\hat{\sigma}}{\sqrt{n}} \tag{3.25}$$

where $\hat{\sigma}$ is a consistent estimator of $\sigma(\theta)$.

3.5 Numerical Illustration

To study the performance of $\hat{L}_S$, random samples of 5000 observations are generated independently 50 times from exponential distributions assuming $\theta_1 = 2$, $\theta_2 = 3$ and $\theta_3 = 4$.Using the generated samples, the Maximum Likelihood estimates of θ_1, θ_2 and θ_3 are obtained. The estimated value of L_S i.e., $\hat{L}_S$ is then obtained using these estimates. Table 2 shows the calculated values of L_S and $\hat{L}_S$ and the corresponding bias which is defined as the difference between the value of $\hat{L}_S$ and L_S.

The performance of $\hat{L}_S$ is measured in terms of Mean Square Error that is defined as Mean Square Error = $\frac{1}{50}\sum(\hat{L}_S - L_S)^2$.

The Mean Square Error value is obtained as 0.002014933. It is to be noted that values of Mean Square Error of any estimator close to zero indicate that the estimator is good. Thus, based on the numerical results, it is reasonable to conclude that the proposed estimator performs well.

Table 2 : Values of L_S and $\hat{L}_S$

Sample . no	L_S	$\hat{L}_S$	Bias	Sample . no	L_S	$\hat{L}_S$	Bias
1	2.418929	2.342763	-0.07617	26	2.418929	2.363453	-0.05548
2	2.418929	2.404658	-0.01427	27	2.418929	2.448545	0.029616
3	2.418929	2.416234	-0.00269	28	2.418929	2.364139	-0.05479
4	2.418929	2.348829	-0.0701	29	2.418929	2.365554	-0.05338
5	2.418929	2.402131	-0.0168	30	2.418929	2.361561	-0.05737
6	2.418929	2.432427	0.013499	31	2.418929	2.40979	-0.00914
7	2.418929	2.37898	-0.03995	32	2.418929	2.400889	-0.01804
8	2.418929	2.383625	-0.0353	33	2.418929	2.41108	-0.00785
9	2.418929	2.392747	-0.02618	34	2.418929	2.393344	-0.02558
10	2.418929	2.400538	-0.01839	35	2.418929	2.386933	-0.032
11	2.418929	2.354361	-0.06457	36	2.418929	2.426333	0.007404
12	2.418929	2.305803	-0.11313	37	2.418929	2.409509	-0.00942
13	2.418929	2.308225	-0.1107	38	2.418929	2.385804	-0.03312
14	2.418929	2.35657	-0.06236	39	2.418929	2.323868	-0.09506
15	2.418929	2.368297	-0.05063	40	2.418929	2.367714	-0.05121
16	2.418929	2.45459	0.035661	41	2.418929	2.413293	-0.00564
17	2.418929	2.430434	0.011505	42	2.418929	2.432742	0.013813
18	2.418929	2.43914	0.020211	43	2.418929	2.396278	-0.02265
19	2.418929	2.39928	-0.01965	44	2.418929	2.40431	-0.01462
20	2.418929	2.438638	0.019709	45	2.418929	2.371189	-0.04774
21	2.418929	2.351422	-0.06751	46	2.418929	2.405219	-0.01371
22	2.418929	2.380057	-0.03887	47	2.418929	2.356025	-0.0629
23	2.418929	2.359503	-0.05943	48	2.418929	2.398101	-0.02083
24	2.418929	2.452838	0.033909	49	2.418929	2.422104	0.003175
25	2.418929	2.44069	0.021761	50	2.418929	2.412362	-0.00657

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