Ageing Properties of Curtate Future Life Time

Kanchan Jain* **and Harmanpreet Singh Kapoor**
Department of Statistics, Panjab University, Chandigarh- 160014,India
Corresponding Author (Email:jaink14@gmail.com)

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ABSTRACT

In insurance sector, for determining the premium of an insured aged x, the insurer is interested not only in the complete future lifetime $T(x)$, but also in the individual's curtate future lifetime $K(x)$ =[$T(x)$]. In this paper, we derive expressions for the reliability measures of $K(x)$ and explore some of its ageing properties.

Keywords: Complete Future Lifetime, Curtate Future Lifetime, IFR, IFRA, DMRL, NBU, NBUE, HNBUE.

1. Introduction

For a new-born child, the age at death is a continuous random variable X with cumulative distribution function (cdf) $F(x)$ and survival function $\bar{F}(x) = P(X > x)$ for $x > 0$. For a non-living object, X represents the age at failure. In actuarial science, $\bar{F}(x)$ is the survival probability that a person/object survives for at least x years and is denoted by $\bar{P}_x$. For X with support on $\{1,2,\ldots\}$, $\bar{P}_x = P(X \geq x)$ and for support $\{0,1,2,\ldots\}$, $\bar{P}_x = P(X > x)$.

The notation (x) is used to denote a life aged x and $T(x)$ denotes the complete future lifetime of (x). According to International Actuarial Notations given in 1949, ${}_tq_x$ is the cdf of $T(x)$ at t and gives the probability that (x) dies within t years. The survival function of $T(x)$ is

${}_tp_x = 1 - {}_tq_x =$ P[$T(x) >$ t] = P[$X > x+t \mid X > x$]= $\dfrac{\bar{P}_{x+t}}{\bar{P}_x}$.

The probability that (x) will die between ages $x+t$ and $x+t+u$ is written as

$$ {}_{t|u}q_x = {}_tp_x - {}_{t+u}p_x = {}_tp_x - {}_tp_x\, {}_up_{x+t} = {}_tp_x\, {}_uq_{x+t} = \frac{\bar{P}_{x+t} - \bar{P}_{x+t+u}}{\bar{P}_x}. \quad (1) $$

For $u = 1$, ${}_{t|1}q_x = {}_{t|}q_x$ and ${}_{0|}q_x = q_x$. The curtate future lifetime of (x) is the discrete number of future years completed by (x) prior to death and is written as
$K(x) = [T(x)]$: the greatest integer less than or equal to $T(x)$.
$K(x)$ is a discrete random variable whereas $T(x)$ is a continuous random variable. One can refer to Neill (1977), Gerber (1990), Bowers et al. (1997), Slud (2001), Borowiak (2003), Zhu (2007) and Dickson et al. (2009) for the above notations and definitions.
In lifetime analysis, an important aspect is to find a lifetime distribution that can adequately describe the ageing behaviour of the concerned life. Lifetimes are continuous in nature and hence many continuous life distributions have been proposed in the literature. On the other hand, discrete failure data arise in several common situations. For example, reports on insured's deaths are collected half yearly or annually and the observations are the number of deaths without specification of the occurrence of events. Sometimes, it is essential to count the complete number of years for which a patient has survived after going through a severe operation. The curtate future lifetime plays a significant role in assurance contracts and discrete life annuities when the benefit is payable at the end of year of death of the claimant/insured. As compared to continuous failure data, interest in discrete analogue arose relatively late. It was only briefly mentioned by Barlow and Proschan (1981). For earlier works on discrete lifetime distributions, one can see Salvia and Bollinger (1982), Xekalaki (1983), Padgett and Spurrier (1985) and Ebrahimi (1986).
Many authors have studied various ageing properties of complete future life time $T(x)$ (termed as residual life in reliability theory) (Ref. Deshpande et al. (1986), Gupta (1987), Lai and Xie (2006) and Nanda et al. (2010)). But the literature is devoid of study of the ageing properties of curtate future lifetime. Hence, we are motivated to explore the ageing properties of $K(x)$, the curtate future lifetime in terms of ageing properties of X. For this purpose, the underlying distribution of X is assumed to be Exponetntial, Weibull, Pareto and Burr.
As we are interested in finding the reliability measures and exploring the ageing properties of $K(x)$, the relevant definitions are listed below (Ref. Johnson, Kemp and Kotz (2005), Lai and Xie (2006) and Sudheesh and Dewan (2009)).

For $\bar{P}_l = P[Y > l]$ and $P_l = P[Y \leq l]$,

1. the discrete failure rate (FR) is the amount of risk associated with an item at time l and is defined as

$$h_y(l) = \frac{P[Y = l]}{P[Y \geq l]} = \frac{\bar{P}_l - \bar{P}_{l+1}}{\bar{P}_l}, \quad l = 0, 1, 2 \ldots \tag{2}$$

2. the discrete reversed failure rate (RFR) is defined as:

$$\tau_y(l) = \frac{P[Y = l]}{P[Y \leq l]} = \frac{\bar{P}_l - \bar{P}_{l+1}}{P_l}, \qquad l = 0, 1, 2 \ldots \tag{3}$$

3. the discrete failure rate average (FRA) is

$$\begin{aligned} \frac{1}{(l+1)} \sum_{j=0}^{l} h_y(j) = & \frac{1}{(l+1)} \sum_{j=0}^{l} \frac{P[Y = j]}{P[Y \geq j]} \\ = & \frac{1}{(l+1)} \sum_{j=0}^{l} \frac{\bar{P}_j - \bar{P}_{j+1}}{\bar{P}_j}, \quad l = 0, 1, 2 \ldots \end{aligned} \tag{4}$$

4. the discrete mean residual life (MRL) is defined as

$$m_y(l) = \sum_{j>l}^{\infty} \frac{P[Y \geq j]}{P[Y \geq l]} = \sum_{j>l}^{\infty} \frac{\bar{P}_j}{\bar{P}_l}, \qquad l = 0, 1, 2 \ldots \tag{5}$$

Some of the discrete ageing properties of Y given by Kelfsjo (1982) and Johnson, Kemp and Kotz (2005) are reproduced below:

1. **Discrete IFR (DFR) :** A discrete distribution with infinite support has a monotonically increasing (decreasing) failure rate with time according as

$$\frac{p_{k+1}}{p_k} > (<) \frac{p_{k+2}}{p_{k+1}} \tag{6}$$

where $p_k = P[Y = k], \quad k = 0, 1, 2 \ldots$

2. **Discrete IFRA (DFRA)**: A discrete lifetime distribution has an increasing or decreasing failure rate average according as

$$\frac{\sum_{j=0}^{k} h_y(j)}{k+1} > (<) \frac{\sum_{j=0}^{k-1} h_y(j)}{k}, \quad k = 1, 2, \ldots \tag{7}$$

or

$$\frac{\sum_{j=0}^{k+1} h_y(j)}{k+2} > (<) \frac{\sum_{j=0}^{k} h_y(j)}{k+1}, \; k = 0, 1, 2, \ldots$$

3. **Discrete IMRL (DMRL):** An increasing (decreasing) mean residual life is determined by

$$\sum_{j>k}^{\infty} \left(\frac{\bar{P}_j}{\bar{P}_k} - \frac{\bar{P}_{j+1}}{\bar{P}_{k+1}} \right) > (<) \, 0, \quad k = 0, 1, 2 \ldots \tag{8}$$

4. **Discrete NBU (NWU):** - The distribution of Y is new better (worse) than used if

$$\bar{P}_{j+k} < (>) \, \bar{P}_j \, \bar{P}_k \quad for \; j, \; k = 0, 1, 2 \ldots \tag{9}$$

5. **Discrete NBUE (NWUE)**: Y has a distribution which is new better (worse) than used in expectation according as:

$$\sum_{j=0}^{\infty} \bar{P}_{j+k} < (>) \, \bar{P}_k \sum_{j=0}^{\infty} \bar{P}_j, \qquad k = 0, 1, 2 \ldots \tag{10}$$

6. **Discrete HNBUE (HNBWE):** Y is harmonically new better (worse) than used in expectation if

$$\sum_{j=k}^{\infty} \bar{P}_j < (>) \, \mu (1 - \frac{1}{\mu})^k, k = 0, 1, 2 \ldots. \tag{11}$$

where $\mu = \sum_{j=0}^{\infty} P[Y > j] = \sum_{j=0}^{\infty} \bar{P}_j$.
The paper is organised as follows:
In Section 2, the expressions for reliability measures of $K(x)$ are derived .The conditions for holding of ageing properties of $K(x)$ have been presented in

Section 3. Section 4 investigates the ageing behaviour of $K(x)$ under the assumption that X follows Exponential, Weibull, Pareto or Burr distribution.

2.Reliability Measures of $K(x)$

In this section, we find the expressions for different reliability measures of

$K(x)$. The probability function of $K(x)$ is given by
$P[K(x)=k] = {}_{k}p_x\ q_{x+k} = {}_{k|}q_x$ (Ref. Bowers et al. (1997)).

1. The failure rate (FR) function of $K(x)$ is given by

$$h_{K(x)}(k) = \frac{P[K(x)=k]}{P[K(x)\geq k]} = \frac{{}_{k|}q_x}{\sum_{l=k}^{\infty} {}_{l|}q_x} \quad k=0,1,2\ldots$$

Using (1), $h_{K(x)}(k) = \dfrac{\bar{P}_{x+k}-\bar{P}_{x+k+1}}{\bar{P}_{x+k}} = q_{x+k}$.

2. The reversed failure rate (RFR) of $K(x)$ can be written as

$$\begin{aligned}\tau_{K(x)}(k) &= \frac{P[K(x)=k]}{P[K(x)\leq k]} = \frac{{}_{k|}q_x}{\sum_{l=0}^{k} {}_{l|}q_x} \\ &= \frac{\bar{P}_{x+k}-\bar{P}_{x+k+1}}{\bar{P}_{x}-\bar{P}_{x+k+1}} = q_{x+k}\left(\frac{\bar{P}_{x+k}}{\bar{P}_{x}-\bar{P}_{x+k+1}}\right) \\ &= q_{x+k}\left(\frac{{}_{k}p_x}{1-{}_{k+1}p_x}\right), k=0,1,2\ldots\end{aligned}$$

3. The failure rate average (FRA) of $K(x)$ is given by

$$\begin{aligned}\frac{1}{k+1}\sum_{j=0}^{k}\frac{P[K(x)=j]}{P[K(x)\geq j]} &= \frac{1}{k+1}\sum_{j=0}^{k}\frac{{}_{j|}q_x}{\sum_{l=j}^{\infty} {}_{l|}q_x} \\ &= \frac{1}{k+1}\sum_{j=0}^{k} q_{x+j}.\end{aligned}$$

4. The mean residual life (MRL) of $K(x)$ is

$$m_{K(x)}(k) = \sum_{j>k}^{\infty} \frac{P[K(x) \geq j]}{P[K(x) \geq k]} = \sum_{j=k+1}^{\infty} \frac{\sum_{l=j}^{\infty} {}_{l|}q_x}{\sum_{l=k}^{\infty} {}_{l|}q_x}$$

$$= \sum_{j=k+1}^{\infty} \frac{\sum_{l=j}^{\infty} (\bar{P}_{x+l} - \bar{P}_{x+l+1})}{\sum_{l=k}^{\infty} (\bar{P}_{x+l} - \bar{P}_{x+l+1})} = \sum_{j=k+1}^{\infty} \frac{\bar{P}_{x+j}}{\bar{P}_{x+k}} = \sum_{j=1}^{\infty} {}_{j}p_{x+k}.$$

For $k=0,\ m_{K(x)}(0) = E[K(x)] = \dot{e}_x$: Curtate expectation of life (Ref. Bowers et al. (1997))
The ageing properties of $K(x)$ are explored in the next section.

3.Ageing properties of $K(x)$

The following theorem specifies the conditions required for possessing of various ageing properties by $K(x)$.

Theorem 1: The distribution of curtate future life time $K(x)$ is

1. IFR (DFR) if $({}_{k+1}p_x - {}_{k+2}p_x)^2 > (<) ({}_{k}p_x - {}_{k+1}p_x)({}_{k+2}p_x - {}_{k+3}p_x), k = 0,1,2,\ldots$;
2. IFRA (DFRA) if $\left(\sum_{j=0}^{k-1} p_{x+j} - k\ p_{x+k}\right) > (<) 0$;
3. DMRL (IMRL) if $\sum_{j=k+1}^{\infty} \frac{{}_{j}p_x}{{}_{k}p_x} - \sum_{j=k+2}^{\infty} \frac{{}_{j}p_x}{{}_{k+1}p_x} > (<) 0$;
4. NBU (NWU) if ${}_{j+k}p_x < (>)\ {}_{j}p_x\ {}_{k}p_x$;
5. NBUE (NWUE) if $\sum_{j=0}^{\infty} {}_{j+k}p_x < (>) {}_{k}p_x \sum_{j=0}^{\infty} {}_{j}p_x$;
6. HNBUE (HNWUE) if $\sum_{j=k}^{\infty} {}_{j}p_x < (>)\ \mu(1-\frac{1}{\mu})^k$ where $\mu = \sum_{j=1}^{\infty} {}_{j}p_x$.

Proof: See the appendix
In the sequel, the distributions of $K(x)$ will be designated as Curtate Distributions.

Remark 1. For exploring the ageing properties NBU (NWU), NBUE (NWUE) and HNBUE (HNWUE) of curtate distributions, we consider the following differences

1. $D_1 = \frac{\bar{P}_{x+j+k}}{\bar{P}_x} - \frac{\bar{P}_{x+j}}{\bar{P}_x}\frac{\bar{P}_{x+k}}{\bar{P}_x}$;
2. $D_2 = \sum_{j=0}^{\infty} \frac{\bar{P}_{x+j+k}}{\bar{P}_x} - \frac{\bar{P}_{x+k}}{\bar{P}_x} \sum_{j=0}^{\infty} \frac{\bar{P}_{x+j}}{\bar{P}_x}$;
3. $D_3 = \sum_{j=k}^{\infty} \frac{\bar{P}_{x+j}}{\bar{P}_x} - \mu(1-\frac{1}{\mu})^k$.

The conclusions are based on positive or negative signs of the above differences. This is proved either mathematically or through figures in case the mathematical form is not tractable.
In the next section, we explore the ageing properties of $K(x)$ when X, the new born's age at death follows some pre-assumed distribution.

4. Ageing Properties

The ageing properties of $K(x)$ are investigated when X, the new born's age at death follows Exponential, Weibull, Pareto or Burr distribution.

4.1. Curtate Exponential Distribution

Let X $\sim$ $Exp(\lambda)$, $\lambda > 0$, then

$$\bar{P}_x = e^{-\lambda x} \quad \text{and} \quad {}_k p_x = \frac{\bar{P}_{x+k}}{\bar{P}_x} = e^{-\lambda k}.$$

Hence the probability mass function (pmf) of $K(x)$ is

$$P[K(x)=k] = \frac{\bar{P}_{x+k} - \bar{P}_{x+k+1}}{\bar{P}_x} = e^{-\lambda k}(1-e^{-\lambda}) \text{ for } \lambda > 0 \text{ and } k = 0,1,2,\ldots$$

Theorem 2**:** If X $\sim$ Exp(λ), then the distribution of curtate future lifetime $K(x)$ is
(i) IFR and DFR (ii) DRFR (iii) IFRA and DFRA (iv) DMRL and IMRL
(v) NBU and NWU (vi) NBUE and NWUE (vii) HNBUE and HNWUE.

Proof: See the appendix.

4.2. Curtate Weibull Distribution

Let $X \sim$ Weibull (α, λ), then $\bar{P}_x = e^{-(\lambda x)^{\alpha}}$ for $\lambda > 0,\ \alpha > 0, x > 0$ gives

$$_{k}p_x = e^{(\lambda x)^{\alpha}} e^{-[\lambda(x+k)]^{\alpha}}.$$

The pmf of $K(x)$ is given by

$$\begin{aligned} P[K(x) = k] &= e^{-\lambda^{\alpha}\left[(x+k)^{\alpha}-(x)^{\alpha}\right]} - e^{-\lambda^{\alpha}\left[(x+k+1)^{\alpha}-(x)^{\alpha}\right]} \\ &= e^{(\lambda x)^{\alpha}}\left[e^{-[\lambda(x+k)]^{\alpha}} - e^{-[\lambda(x+k+1)]^{\alpha}}\right]. \end{aligned} \tag{12}$$

From (2), (3), (4) and (5) FR, RFR, FRA and MRL for Curtate Weibull distribution are derived as :

$\text{FR} = 1 - e^{-\lambda^{\alpha}\left[(x+k+1)^{\alpha}-(x+k)^{\alpha}\right]}$,

$$\text{RFR} = \frac{e^{-\lambda^{\alpha}[(x+k)^{\alpha}-(x)^{\alpha}]} - e^{-\lambda^{\alpha}[(x+k+1)^{\alpha}-(x)^{\alpha}]}}{1 - e^{-\lambda^{\alpha}[(x+k+1)^{\alpha}-(x)^{\alpha}]}},$$

$$\text{FRA} = \frac{1}{k+1}\sum_{j=0}^{k}\left(1 - e^{-\lambda^{\alpha}\left[(x+j+1)^{\alpha}-(x+j)^{\alpha}\right]}\right)$$

$$\approx 1 - \sum_{j=0}^{k} e^{-\alpha\lambda^{\alpha}(x+j)^{\alpha-1} + \dfrac{\alpha(\alpha-1)\lambda^{\alpha}(x+j)^{\alpha-2}}{2}} \quad \text{when } \alpha \text{ is an integer}$$

,

$$\text{MRL} = \sum_{j=k}^{\infty} e^{-\lambda^{\alpha}\left[(x+j+1)^{\alpha}-(x+k)^{\alpha}\right]}$$

Theorem 3: If $X \sim$ Weibull(α, λ) , then the distribution of curtate future lifetime $K(x)$ is
(i) IFR (DFR) for $\alpha > (<) 1$ and any λ ;
(ii) DRFR if

(a) $\alpha > 1$ and $\lambda > \left(\dfrac{\alpha - 1}{\alpha(x+k)^{\alpha}}\right)^{\dfrac{1}{\alpha}}$;
(b) $\alpha < 1$ and all $\lambda > 0$.
(iii) IFRA (DFRA) for $\alpha > (< 1)$ and all $\lambda > 0$;

Proof See the appendix
Tables $1-3$ (given in the appendix) display the values of failure rate, reversed failure rate and failure rate average for different values of α and λ

and for $x = 20, 30, 40, 50$. The tables are based on hypothetical data which can arise in real life situations.
On the basis of Table $1-3$, it can be concluded that

- for each $\lambda > 0$, and
  (i) $0 < \alpha < 1$, FR decreases in k as well as x.
  (ii) $\alpha > 1$, FR increases in k as well as x.

- RFR
  (i) decreases if $0 < \alpha < 1$ and $\lambda > 0$ and
  (ii) increases if $\alpha > 1$ and $\lambda > \left(\frac{\alpha - 1}{\alpha(x+k)^{\alpha}}\right)^{\frac{1}{\alpha}}$

- similarly for $\lambda > 0$, FRA decreases in k and x if $0 < \alpha < 1$ and increases in k as well as x if $\alpha > 1$.

The mean residual life function is plotted in Figure $1a - 1b$
Figures $2-4$ plot the following differences
$D_1 = e^{-\lambda^{\alpha}((x+j+k)^{\alpha} - x^{\alpha})} - e^{-\lambda^{\alpha}((x+j)^{\alpha} + (x+k)^{\alpha} - 2x^{\alpha})}$;
$D_2 = \sum_{j=0}^{\infty} e^{-\lambda^{\alpha}[(x+j+k)^{\alpha} - x^{\alpha})} - e^{-\lambda^{\alpha}[(x+k)^{\alpha} - x^{\alpha})} \sum_{j=0}^{\infty} e^{-\lambda^{\alpha}[(x+j)^{\alpha} - x^{\alpha})}$;
and
$$D_3 = \sum_{j=k}^{\infty} e^{-\lambda^{\alpha}\left[(x+j)^{\alpha} - x^{\alpha}\right)} - \sum_{j=1}^{\infty} e^{-\lambda^{\alpha}\left[(x+j)^{\alpha} - x^{\alpha}\right)} \left(1 - \frac{1}{\sum_{j=1}^{\infty} e^{-\lambda^{\alpha}\left[(x+j)^{\alpha} - x^{\alpha}\right)}}\right)^{k}.$$

From Figures $1a-4$, it can be concluded that for any $\lambda > 0$, $K(x)$ is

- DMRL (IMRL) for $\alpha > (<) 1$;
- NBU (NWU) for all $\alpha > (<) 1$;
- NBUE (NWUE) for all $\alpha > (<) 1$;
- HNBUE (HNWUE) for $\alpha > (<) 1$.

Remark (2) : Exponential Distribution is a special case of Weibull distribution for $\alpha = 1$, and in that case both the distributions exhibit the same ageing behaviour.

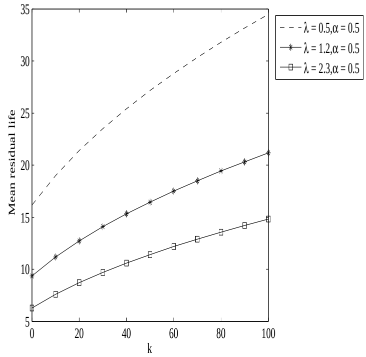

Figure 1a: MRL for $\alpha < 1$

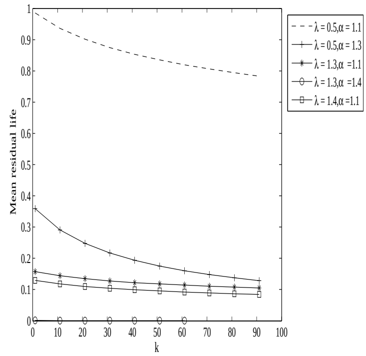

Figure 1b : MRL for $\alpha > 1$

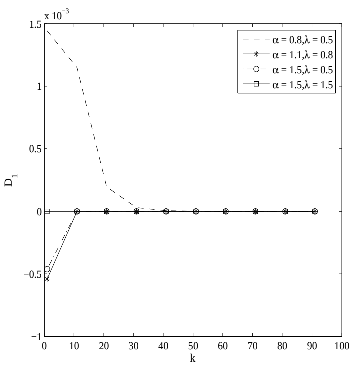

Figure 2: D_1 for Curtate Weibull Distri.

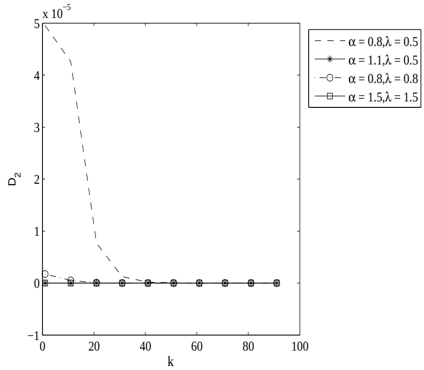

Figure 3: D_2 for Curtate Weibull Distri.

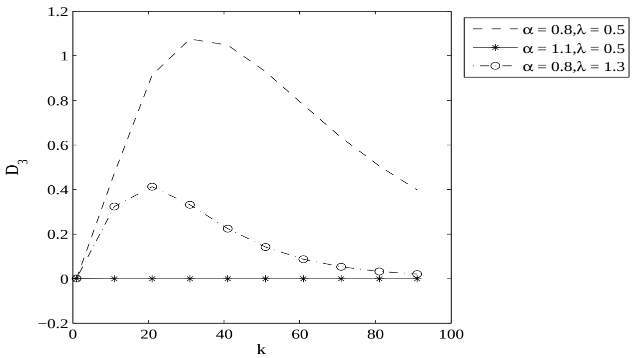

Figure 4: D_3 for Curtate Weibull Distribution

4.3. Curtate Pareto distribution

If $X \sim$ Pareto(θ), then $\bar{P}_x = \frac{1}{x^\theta}$, $\theta > 0, x > 0$.

This gives ${}_k\mathrm{p}_x = \left(\frac{x}{x+k}\right)^\theta$, $x > 0$. Hence,

$$
\begin{aligned}
P[K(x) = k] &= \left(\frac{x}{x+k}\right)^\theta - \left(\frac{x}{x+k+1}\right)^\theta \\
&= \left(1+\frac{k}{x}\right)^{-\theta} - \left(1+\frac{k+1}{x}\right)^{-\theta} \quad \text{for } k = 0,1,2\ldots
\end{aligned}
$$

Using (2) - (5), FR, RFR, FRA and MRL for Curtate Pareto distribution are derived as:

FR= $1 - \frac{(x+k)^\theta}{(x+k+1)^\theta}$,

RFR = $\left(\frac{(x+k+1)^\theta - (x+k)^\theta}{(x+k)^\theta - (x)^\theta}\right)\left(\frac{x^\theta}{(x+k+1)^\theta}\right)$,

FRA = $\frac{1}{k+1}\sum_{j=0}^{k}\left(1 - \frac{(x+j)^\theta}{(x+j+1)^\theta}\right)$

$$\approx -k - \theta\left(\psi(0,x) - \psi(0,1+k+x)(1+\theta)(\psi(1,x) - \psi(1,1+k+x))\right),$$

when θ is an integer and $\psi(n,x)$, the digamma function denotes the n^{th}derivative of gamma function.

MRL = $\sum_{j=k+1}^{\infty}\frac{(k+x)^\theta}{(j+x)^\theta}$ for θ, an integer.

Theorem 4: If X $\sim$ Pareto(θ), then the distribution of curtate future lifetime $K(x)$ is (i) DFR (ii) DRFR (iii) DFRA (iv) IMRL (v) NWU (vi) NWUE.

Proof See the appendix.

For concluding about HNWUE, we consider

$$D_3 = \sum_{j=k}^{\infty}\left(\frac{x}{j+x}\right)^\theta - \sum_{j=1}^{\infty}\left(\frac{x}{j+x}\right)^\theta\left(1 - \frac{1}{\sum_{j=1}^{\infty}\left(\frac{x}{j+x}\right)^\theta}\right)^\theta.$$

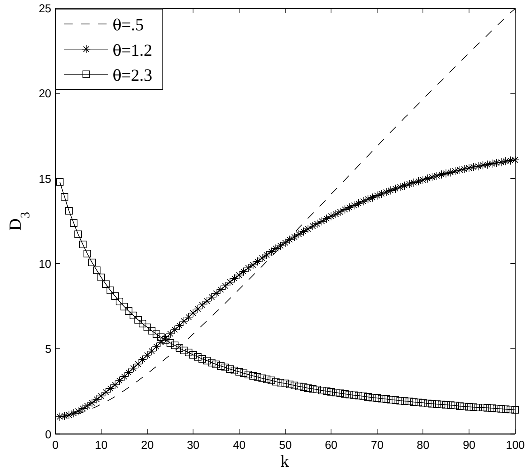

Figure 5 : D_3 for Curtate Pareto Distribution

As it is difficult to decide the sign of D_3, we try to conclude by plotting D_3 in Figure 5.
Since $D_3 > 0$ for all θ, hence the Curtate Pareto distribution is HNWUE.
The values of FR, RFR, FRA and MRL have been displayed in Tables $4-7$ for curtate Pareto distribution. From these tables, we can conclude that

- the values of FR decrease as initial age x and k increase for all $\theta > 0$.
- the RFR is a decreasing function of x and k.
- FRA values show a decreasing trend as the values of x and k increase for all values of θ.
- MRL is an increasing function of initial age x and k for all θ.

4.4. Curtate Burr Distribution

If $X \sim \text{Burr}(\theta, c)$, then $\bar{P}_x = \dfrac{1}{(1+x^c)^\theta}$, $\theta > 0, c > 0$ and $x > 0$.

This give ${}_k\text{p}_x = \left(\frac{1+x^c}{1+(x+k)^c}\right)^{\theta}$.
Hence

$$P[K(x)=k] = \left(\frac{1+x^c}{1+(x+k)^c}\right)^{\theta} - \left(\frac{1+x^c}{1+(x+k+1)^c}\right)^{\theta}$$
$$= (1+x^c)^{\theta}\left(\frac{1}{(1+(x+k)^c)^{\theta}} - \frac{1}{(1+(x+k+1)^c)^{\theta}}\right).$$

Using (2) - (5), the FR, RFR, FRA and MRL for Curtate Burr distribution are

$$\text{FR} = 1 - \frac{(1+(x+k)^c)^{\theta}}{(1+(x+k+1)^c)^{\theta}},$$
$$\text{RFR} = \left(\frac{(1+(x+k+1)^c)^{\theta} - (1+(x+k)^c)^{\theta}}{(1+(x+k+1)^c)^{\theta} - (1+x^c)^{\theta}}\right)\left(\frac{(1+x^c)^{\theta}}{(1+(x+k)^c)^{\theta}}\right),$$
$$\text{FRA} = \frac{1}{k+1}\Sigma_{j=0}^{k}\left(1 - \frac{(1+(x+j)^c)^{\theta}}{(1+(x+j+1)^c)^{\theta}}\right)$$
$$\approx 1 - \frac{1}{k+1}\Sigma_{j=0}^{k}\left(1 + \frac{c(x+j)^{c-1} + c(c-1)(x+j)^{c-2}}{(1+(x+j)^c)}\right)^{-\theta},$$
$$\text{MRL} = \Sigma_{j=k+1}^{\infty}\frac{(1+(x+k)^c)^{\theta}}{(1+(x+j)^c)^{\theta}}$$
$$\approx \Sigma_{j=1}^{\infty}\frac{1}{(1+jc(x+k)^{c-1} + \frac{jc(c-1)(x+k)^{c-2}}{2})}$$

where θ and c are integers.

For Curtate Burr distribution, we conclude about the ageing properties by looking at the plots of different reliability measures as shown in Figures $6-9$. Figures $10-12$ plot the differences

$$D_1 = \left(\frac{1+x^c}{1+(x+j+k)^c}\right)^{\theta} - \left(\frac{1+x^c}{1+(x+j)^c}\right)^{\theta}\left(\frac{1+x^c}{1+(x+k)^c}\right)^{\theta};$$
$$D_2 = \Sigma_{j=0}\left(\frac{1+x^c}{1+(x+j+k)^c}\right)^{\theta} - \left(\frac{1+x^c}{1+(x+k)^c}\right)^{\theta}\Sigma_{j=0}\left(\frac{1+x^c}{1+(x+j)^c}\right)^{\theta};$$

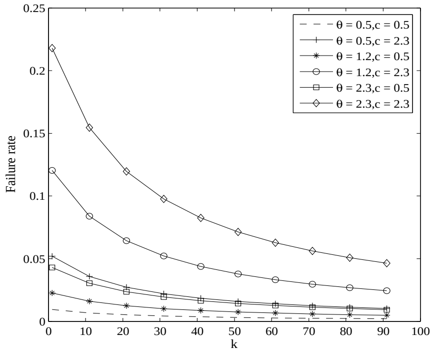

Figure 6: Failure Rate

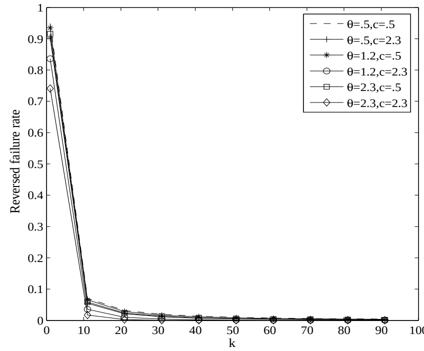

Figure 7: Reversed failure Rate

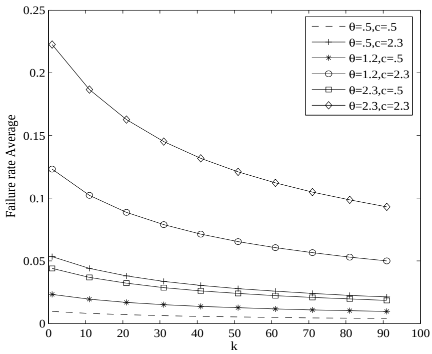

Figure 8: Failure Rate Average

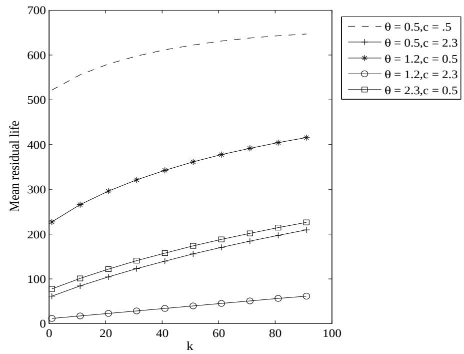

Figure 9: Mean Residual life

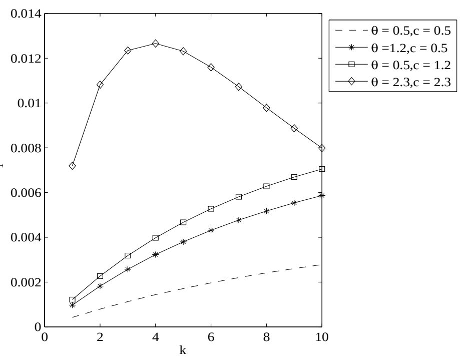

Figure 10: D_1 for Curtate Burr Distri.

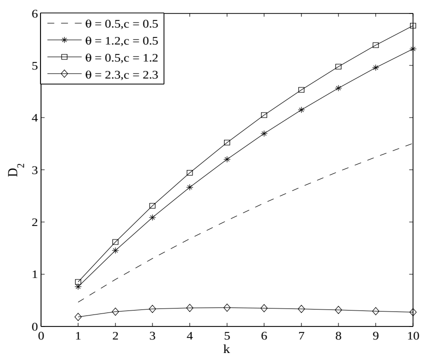

Figure 11: D_2 for Curtate Burr Distri.

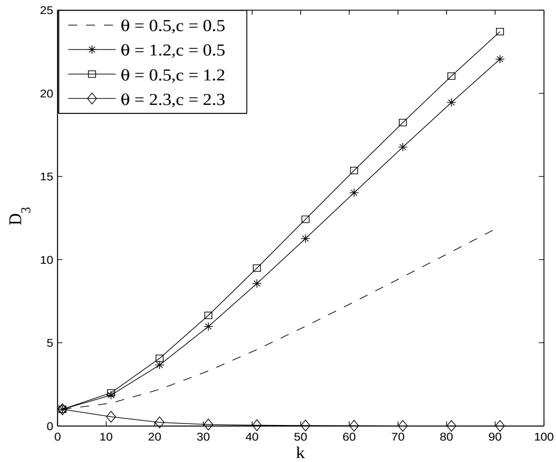

Figure 12: D_3 for Curtate Burr Distribution

$$D_3 = \sum_{j=k}\left(\frac{1+x^c}{1+(x+j)^c}\right)^{\theta} - \sum_{j=1}^{\infty}\left(\frac{1+x^c}{1+(x+j)^c}\right)^{\theta}\left(1-\frac{1}{\sum_{j=1}^{\infty}\left(\frac{1+x^c}{1+(x+j)^c}\right)^{\theta}}\right)^{\theta}.$$

Figures $6-12$ help us to conclude that $K(x)$ is DFR, DRFR, DFRA and IMRL for different parametric combinations. It can also be concluded that for all θ and c, $K(x)$ is NWU, NWUE and HNWUE.

On the basis of the above discussion , it is concluded that the considerded ageing properties of the distribution of X are preserved by the curtate future lifetime distributions.

5. Conclusions

In this paper, we have studied the ageing properties of the curtate future lifetime distributions. It is shown that the ageing properties viz IFR (DFR), IFRA (DFRA), DMRL(IMRL), NBU (NWU), NBUE (NWUE) and HNBUE (HNWUE) are preserved by the curtate future lifetime distributions, when the new born's age at death follows Exponential, Weibull, Pareto and Burr Distributions.

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Appendix

Proof of Theorem 1:

(i) Using (6), $K(x)$ will have an increasing (decreasing) failure rate if

$$\frac{P[K(x)=k+1]}{P[K(x)=k]} > (<) \frac{P[K(x)=k+2]}{P[K(x)=k+1]}, \quad k=0,1,2,3,\ldots$$

that is, if

$$\frac{\bar{P}_{x+k+1}-\bar{P}_{x+k+2}}{\bar{P}_{x+k}-\bar{P}_{x+k+1}} > (<) \frac{\bar{P}_{x+k+2}-\bar{P}_{x+k+3}}{\bar{P}_{x+k+1}-\bar{P}_{x+k+2}}$$

This gives

$$(\bar{P}_{x+k+1})^2+(\bar{P}_{x+k+2})^2-\bar{P}_{x+k+1}\bar{P}_{x+k+2}-\bar{P}_{x+k}\bar{P}_{x+k+2}+\bar{P}_{x+k}\bar{P}_{x+k+3}-\bar{P}_{x+k+1}\bar{P}_{x+k+3} > (<) 0.$$

Alternatively, $K(x)$ has IFR (DFR) if

$$\left({}_{k+1}p_x - {}_{k+2}p_x\right)^2 > (<) \left({}_{k}p_x - {}_{k+1}p_x\right)\left({}_{k+2}p_x - {}_{k+3}p_x\right), k=0,1,2,3,\ldots$$

Using, ${}_{k+1}p_x = {}_{k}p_x \; {}_{1}p_{x+k}$ and ${}_{k+2}p_x = {}_{k}p_x \; {}_{2}p_{x+k}$,
this is equivalent to

$$\left({}_{1}p_{x+k} - {}_{2}p_{x+k}\right)^2 > (<) \left(1 - {}_{1}p_{x+k}\right)\left({}_{2}p_{x+k} - {}_{3}p_{x+k}\right).$$

This gives

$$\frac{{}_{1}p_{x+k} - {}_{2}p_{x+k}}{1 - p_{x+k}} > (<) \frac{{}_{2}p_{x+k} - {}_{3}p_{x+k}}{{}_{1}p_{x+k} - {}_{2}p_{x+k}},$$

that is

$$\frac{{}_{j}p_{x+k} - {}_{j+1}p_{x+k}}{{}_{j-1}p_{x+k} - {}_{j}p_{x+k}} \text{ is non}-\text{increasing (non }-\text{decreasing) in j.}$$

(ii) $K(x)$ will have an increasing or decreasing failure rate average (IFRA or DFRA) if $\frac{\sum_{j=0}^{k} h_{K(x)}(j)}{k+1}$ is increasing (decreasing) in k.

This happens if

$$\frac{\sum_{j=0}^{k} h_{K(x)}(j)}{k+1} > (<) \frac{\sum_{j=0}^{k-1} h_{K(x)}(j)}{k}, k = 0,1,2,3\ldots$$

that is , if

$$\frac{\sum_{j=0}^{k} (\bar{P}_{x+j} - \bar{P}_{x+j+1})}{(k+1)\bar{P}_{x+j}} > (<) \frac{\sum_{j=0}^{k-1} (\bar{P}_{x+j} - \bar{P}_{x+j+1})}{(k)\bar{P}_{x+j}}.$$

This means that

$$\left(\sum_{j=0}^{k-1} \frac{\bar{P}_{x+j+1}}{\bar{P}_{x+j}} - k \frac{\bar{P}_{x+k+1}}{\bar{P}_{x+k}} \right) > (<) 0.$$

Since ${}_k\mathrm{p}_x = \dfrac{\bar{P}_{x+k}}{\bar{P}_x}$, the above inequality can be expressed as

$$\left(\sum_{j=0}^{k-1} p_{x+j} - k\ p_{x+k} \right) > (<) 0$$

or

$$\left(\sum_{j=0}^{k} p_{x+j} - (k+1) p_{x+k+1} \right) > (<) 0.$$

It means that the sum of probabilities of survival upto $(x+k)^{th}$ year is greater or less than $k+1$ times the survival probability of life aged $(x+k+1)$ during the next one year.
(iii) Decreasing (Increasing) mean residual life DMRL (IMRL) of $K(x)$ is indicated if

$$\sum_{j>k}^{\infty} \left(\frac{P[K(x) \geq j]}{P[K(x) \geq k]} - \frac{P[K(x) \geq j+1]}{P[K(x) \geq k+1]} \right) < (>) 0.$$

This holds if

$$\sum_{j=k+1}^{\infty} \frac{\bar{P}_{x+j}}{\bar{P}_{x+k}} - \sum_{j=k+2}^{\infty} \frac{\bar{P}_{x+j}}{\bar{P}_{x+k+1}} < (>) 0.$$

Alternatively,

$$\sum_{j=k+1}^{\infty} \frac{{}_j p_x}{{}_k p_x} - \sum_{j=k+2}^{\infty} \frac{{}_j p_x}{{}_{k+1} p_x} < (>)\ 0,$$

that is

$$\sum_{j=k+1}^{\infty} \frac{{}_jp_x}{{}_kp_x} \text{ is non}-\text{decreasing or non}-\text{increasing in } j > 0.$$

(iv) The distribution of $K(x)$ will be new better (worse) than used (NBU (NWU)) if the following condition is satisfied

$$P[K(x) = j+k] < (>) P[K(x) = j] P[K(x) = k]$$

that is

$$\frac{\bar{P}_{x+j+k}}{\bar{P}_x} < (>) \frac{\bar{P}_{x+j}}{\bar{P}_x} \frac{\bar{P}_{x+k}}{\bar{P}_x}$$

which happens if

$${}_{j+k}p_x < (>) \, {}_jp_x \, {}_kp_x \quad \text{for } j,k = 0,1,2,3,\ldots$$

This means that $K(x)$ is NBU (NWU) if the probability that (x) survives for $(j+k)$ years is less (greater) than the product of survival probabilities of (x) till j and k years.
(v) The distribution of $K(x)$ will have a new better (worse) than used in expectation (NBUE (NWUE)) if the following condition is satisfied

$$\sum_{j=0}^{\infty} P[K(x) \geq j+k] < (>) P[K(x) \geq k] \sum_{j=0}^{\infty} P[K(x) \geq j]$$

that is,

$$\sum_{j=0}^{\infty} \frac{\bar{P}_{x+j+k}}{\bar{P}_x} < (>) \frac{\bar{P}_{x+k}}{\bar{P}_x} \sum_{j=0}^{\infty} \frac{\bar{P}_{x+j}}{\bar{P}_x}$$

or

$$\sum_{j=0}^{\infty} {}_{j+k}p_x < (>) {}_kp_x \sum_{j=0}^{\infty} {}_jp_x, \text{ using } {}_{k+j}p_x = {}_kp_x \, {}_jp_{x+k}.$$

(vi) The distribution of K(x) will be Harmonically new better (worse) than used in expectation (HNBUE (HNWUE)) if

$$\sum_{j=k}^{\infty} P[K(x) \geq j] < (>) \mu \left(1 - \frac{1}{\mu}\right)^k,$$

that is,

$$\sum_{j=k}^{\infty} \frac{\bar{P}_{x+j}}{\bar{P}_x} < (>) \mu (1 - \frac{1}{\mu})^k$$

or

$$\sum_{j=k}^{\infty} {}_j p_x < (>) \mu (1 - \frac{1}{\mu})^k$$

where $\mu = \sum_{j=1}^{\infty} \frac{\bar{P}_{x+j}}{\bar{P}_x} = \sum_{j=1}^{\infty} {}_j p_x$.

Proof of Theorem 2

Using (2) - (5) the FR, RFR, FRA and MRL for Curtate Exponential distribution are

$$\left(1 - e^{-\lambda}\right), \; \frac{e^{-\lambda k}\left(1 - e^{-\lambda}\right)}{\left(1 - e^{-\lambda(k+1)}\right)}, (1 - e^{-\lambda}), \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \quad \text{respectively.}$$

(i), (iii) and (iv) follow since FR, FRA and MRL are independent of k.
(ii) follows since

$$\frac{d}{dk}\left(\frac{e^{-\lambda k}\left(1 - e^{-\lambda}\right)}{\left(1 - e^{-\lambda(k+1)}\right)}\right) = \frac{-\lambda e^{-\lambda k}}{\left(1 - e^{-\lambda(k+1)}\right)^2} < 0.$$

(v) Since

$$\frac{\bar{P}_{x+j+k}}{\bar{P}_x} = e^{-\lambda(j+k)} = \frac{\bar{P}_{x+j}}{\bar{P}_x} \frac{\bar{P}_{x+k}}{\bar{P}_x},$$

hence Curtate Exponential is both NBU and NWU.
(vi) The curtate exponential distribution is both NBUE and NWUE. Since

$$\sum_{j=0}^{\infty} \frac{\bar{P}_{x+j+k}}{\bar{P}_x} = \frac{e^{-\lambda k}}{1 - e^{-\lambda}} = \frac{\bar{P}_{x+k}}{\bar{P}_x} \sum_{j=0}^{\infty} \frac{\bar{P}_{x+j}}{\bar{P}_x}.$$

(vii) The ageing properties HNBUE and HNWUE follow since

$$\sum_{j=k}^{\infty} \frac{\bar{P}_{x+j}}{\bar{P}_x} = \frac{e^{-\lambda k}}{1 - e^{-\lambda}} = \mu (1 - \frac{1}{\mu})^k, \quad k = 0, 1, 2, 3 \ldots$$

Proof of Theorem 3:
Let

$$f(k) = e^{-\lambda^{\alpha}\left\{(x+k+1)^{\alpha}-(x+k)^{\alpha}\right\}}.$$

This gives

$$\frac{df(k)}{dk} = -\alpha\lambda^{\alpha}\left\{(x+k+1)^{\alpha}-(x+k)^{\alpha}\right\}e^{-\lambda^{\alpha}\left\{(x+k+1)^{\alpha}-(x+k)^{\alpha}\right\}}$$

$$< (>)\, 0 \text{ for } \alpha > (<)\, 1.$$

Hence $f(k)$ is decreasing for $\alpha > 1$ and increasing for $\alpha < 1$.
(i) Since FR$= 1 - f(k)$, hence FR is increasing for $\alpha > 1$ and decreasing for $\alpha < 1$.
(ii) Consider $\frac{d(RFR)}{dk} = \frac{d}{dk}\left[\frac{e^{-\lambda^{\alpha}[(x+k)^{\alpha}-(x)^{\alpha}]} - e^{\lambda^{\alpha}[(x+k+1)^{\alpha}-(x)^{\alpha}]}}{1-e^{-\lambda^{\alpha}[(x+k+1)^{\alpha}-(x)^{\alpha}]}}\right]$

Numerator of $\frac{d(RFR)}{dk} = \left(1-e^{\lambda^{\alpha}\left[(x+k+1)^{\alpha}-x^{\alpha}\right]}\right)$

$$\left(\alpha\lambda^{\alpha}(x+k+1)^{\alpha-1}e^{-\lambda^{\alpha}\left[(x+k+1)^{\alpha}-x^{\alpha}\right]} - \alpha\lambda^{\alpha}(x+k)^{\alpha-1}e^{-\lambda^{\alpha}\left[(x+k)^{\alpha}-x^{\alpha}\right]}\right)$$

$$-\left(e^{-\lambda^{\alpha}\left[(x+k)^{\alpha}-x^{\alpha}\right]} - e^{-\lambda^{\alpha}\left[(x+k+1)^{\alpha}-x^{\alpha}\right]}\right)\left(\alpha\lambda^{\alpha}(x+k+1)^{\alpha-1}e^{-\lambda^{\alpha}\left[(x+k+1)^{\alpha}-x^{\alpha}\right]}\right) < 0$$

if

$$(x+k+1)^{\alpha-1}e^{-(\lambda(x+k+1))^{\alpha}} < (x+k)^{\alpha-1}e^{-(\lambda(x+k))^{\alpha}}.$$

This happens if $l(k) = (x+k)^{\alpha-1}e^{-(\lambda(x+k))^{\alpha}}$ is a decreasing function of k.

Consider $\frac{dl(k)}{dk} = (x+k)^{\alpha-2}e^{-(\lambda(x+k))^{\alpha}}\left\{(\alpha-1)-\alpha\lambda^{\alpha}(x+k)^{\alpha}\right\}$.

For $\alpha < 1$ and $\lambda > 0$, $\frac{dl(k)}{dk} < 0$ and

for $\alpha > 1$, $\frac{dl(k)}{dk} < 0$ if $(\alpha-1) < \alpha\lambda^{\alpha}(x+k)^{\alpha}$,that is if $\lambda > \left(\frac{\alpha-1}{\alpha(x+k)^{\alpha}}\right)^{\frac{1}{\alpha}}$.

Denominator of $\frac{d(RFR)}{dk} = \left(1-e^{-\lambda^{\alpha}(x+k+1)^{\alpha}-(x)^{\alpha}]}\right)^2 > 0.$

Therefore $\dfrac{d\,(RFR)}{dk} < 0$

(a) for α <1 and $\lambda > 0$;

(b) for $\alpha > 1$ and $\lambda > \left(\dfrac{\alpha-1}{\alpha(x+k)^{\alpha}}\right)^{\dfrac{1}{\alpha}}$.

This gives the required conclusion.
(iii) For $\alpha < (>)\,1$, $\quad \sum_{j=0}^{k}(1-\mathrm{e}^{-\lambda^{\alpha}[(\mathrm{x+j+1})^{\alpha}-(\mathrm{x+j})^{\alpha}]}$,

being the sum of (k+1) decreasing (increasing) functions, is a (an) decreasing (increasing) function.

For $\alpha < 1$, FRA is the product of two positive decreasing functions. So $K\,(x)$ has DFRA.

For $\alpha > 1$, it is difficult to conclude as FRA is the product of a decreasing and an increasing function. By considering few parametric combinations for $\alpha > 1$ and $\lambda > 0$, the values of FRA are given evaluated in Table 3. It is obvious from values in Table 3 that FRA is decreasing (increasing) for $\alpha < (>)\,1$ and all $\lambda > 0$.

Proof of Theorem 4

Let $f\,(k) = \dfrac{(x+k)^{\theta}}{(x+k+1)^{\theta}}$.

$$\frac{df(k)}{dk} = \frac{\theta(x+k)^{\theta-1}(x+k+1)^{\theta} - \theta(x+k+1)^{\theta-1}(x+k)^{\theta}}{(x+k+1)^{2\theta}}$$

$$= \frac{\theta(x+k)^{\theta-1}(x+k+1)^{\theta-1}}{(x+k+1)^{2\theta}} = \frac{\theta(x+k)^{\theta-1}}{(x+k+1)^{\theta+1}} > 0.$$

Consider $g\,(k) = \left(\dfrac{(x+k+1)^{\theta}-(x+k)^{\theta}}{(x+k+1)^{\theta}-x^{\theta}}\right)\left(\frac{x^{\theta}}{(x+k)^{\theta}}\right)$

$$\frac{dg(k)}{dk} = -\theta\left[\frac{\left((x+k+1)^{\theta-1}(x+k)^{\theta-1} - (k+1)(x+k+1)^{\theta-1}x^{\theta-1} + k(x+k)^{\theta}x^{\theta-1}\right)}{\left((x+k)^{\theta}-x^{\theta}\right)^{2}}\right.$$

$$\frac{x^{\theta}}{(x+k)^{\theta}}+\frac{(x+k+1)^{\theta}-(x+k)^{\theta}}{(x+k+1)^{\theta}-x^{\theta}}\left(\frac{x^{\theta}}{(x+k)^{\theta+1}}\right)\Bigg]<0$$

Hence $f(k)$ is increasing and $g(k)$ is decreasing in k for $k\geq 0$.

(i) Since FR $=1-f(k)$, hence FR is a decreasing function of k for $k\geq 0$.

(ii) RFR $=g\left(k\right)$ implies that Curtate Pareto is DRFR.

(iii) FRA $=\dfrac{1}{k+1}\sum_{j=0}^{k}\left(1-f\left(j\right)\right)$.

Since each $f(j)$ is increasing for j=0,1,... k, hence $\sum_{j=0}^{k}\left(1-f\left(j\right)\right)$, being the sum of k decreasing functions will be decreasing. Since $\dfrac{1}{k+1}$ is decreasing in k hence FRA being the product of two decreasing functions is decreasing in k.

(iv) Since practically k can take values only upto 100 years, hence we write

$$\text{MRL}=\sum_{j=k+1}^{100}\frac{(x+k)^{\theta}}{(x+j)^{\theta}}$$
$$=(x+k)^{\theta}\left(\frac{1}{(x+k+1)^{\theta}}+\frac{1}{(x+k+2)^{\theta}}+\dots\frac{1}{(x+100)^{\theta}}\right).$$

$$\frac{d}{dk}\left(\frac{(x+k)^{\theta}}{(x+k+1)^{\theta}}\right)=\frac{\theta(x+k)^{\theta-1}}{(x+k+1)^{\theta+1}}>0 \text{ for all } \theta.$$

This gives that $\dfrac{(x+k)^{\theta}}{(x+k+1)^{\theta}}$ is an increasing function of k.

It can be shown similarly that, $\dfrac{(x+k)^{\theta}}{(x+k+2)^{\theta}}$ is an increasing function of k.

Hence for θ, $\dfrac{(x+k)^{\theta}}{(x+j)^{\theta}}$ for $j=k+1,k+2,\dots 100$ are increasing functions of k. This implies that MRL, being the sum of a finite number of increasing (non-decreasing) functions of k, is increasing for all θ.

(v) $K(x)$ will be NWU if $D_1=\left(\dfrac{x}{x+k+j}\right)^{\theta}-\dfrac{x^{2\theta}}{(x+j)^{\theta}(x+k)^{\theta}}>0;$

that is if $\dfrac{1}{x^{\theta}(x+j+k)^{\theta}}>\dfrac{1}{(x+j)^{\theta}(x+k)^{\theta}}$ which is true for all values of θ

and $j,k \geq 0$.

(vi) Similarly consider the difference
$D_2 = \sum_{j=k}^{\infty}\left(\frac{x}{x+j+k}\right)^{\theta} - \left(\frac{x}{x+k}\right)^{\theta}\sum_{j=k}^{\infty}\left(\frac{x}{x+j}\right)^{\theta}$.
From (v), we have $\frac{x^{\theta}}{(x+j+k)^{\theta}} - \frac{x^{2\theta}}{(x+j)^{\theta}(x+k)^{\theta}} > 0$ for all $\theta > 0$ and $k \geq 0$.
This implies that $\sum_{j=k}^{\infty}\frac{x^{\theta}}{(x+j+k)^{\theta}} > \sum_{j=k}^{\infty}\frac{x^{2\theta}}{(x+j)^{\theta}(x+k)^{\theta}}$

Hence $D_2 > 0$. This implies that $K(x)$ is NWUE for all θ.
Tables $1-9$ displaying values of FR, RFR, FRA and MRL for different curtate distribuitons are presented now.

Table 1: Failure Rate of curtate Weibull Distribution

x	k	$\alpha = 0.5$ $\lambda = 0.5$	$\alpha = 0.5$ $\lambda = 1.2$	$\alpha = 1.2$ $\lambda = 0.5$	$\alpha = 1.2$ $\lambda = 1.2$
20	10	0.0620	0.0944	0.6447	0.9481
	20	0.0540	0.0825	0.6655	0.9563
	30	0.0485	0.0742	0.6816	0.9621
	40	0.0444	0.0680	0.6947	0.9664
	50	0.0412	0.0632	0.7058	0.9697
30	10	0.0540	0.0825	0.6655	0.9563
	20	0.0485	0.0742	0.6816	0.9621
	30	0.0444	0.0680	0.6947	0.9664
	40	0.0412	0.0632	0.7058	0.9697
	50	0.0386	0.0592	0.7153	0.9725
40	10	0.0485	0.0742	0.6816	0.9621
	20	0.0444	0.0680	0.6947	0.9664
	30	0.0412	0.0632	0.7058	0.9697
	40	0.0386	0.0592	0.7153	0.9725
	50	0.0365	0.0559	0.7237	0.9747
50	10	0.0444	0.0680	0.6947	0.9664
	20	0.0412	0.0632	0.7058	0.9697
	30	0.0386	0.0592	0.7153	0.9725
	40	0.0365	0.0559	0.7237	0.9747
	50	0.0347	0.0532	0.7311	0.9766

Table 2: Reversed Failure Rate of curtate Weibull Distribution

x	k	$\alpha = 0.5$ $\lambda = 0.5$	$\alpha = 0.5$ $\lambda = 1.2$	$\alpha = 1.2$ $\lambda = 0.5$	$\alpha = 1.2$ $\lambda = 1.2$
	10	0.0599	0.0470	0.0000	0.0000
	20	0.0200	0.0125	0.0000	0.0000
20	30	0.0092	0.0046	0.0000	0.0000
	40	0.0049	0.0019	0.0000	0.0000
	50	0.0028	0.0009	0.0000	0.0000
	10	0.0659	0.0539	0.0000	0.0000
	20	0.0233	0.0157	0.0000	0.0000
30	30	0.0112	0.0062	0.0000	0.0000
	40	0.0061	0.0028	0.0000	0.0000
	50	0.0036	0.0014	0.0000	0.0000
	10	0.0698	0.0586	0.0000	0.0000
	20	0.0257	0.0182	0.0000	0.0000
40	30	0.0127	0.0075	0.0000	0.0000
	40	0.0072	0.0036	0.0000	0.0000
	50	0.0044	0.0018	0.0000	0.0000
	10	0.0727	0.0621	0.0000	0.0000
	20	0.0275	0.0202	0.0000	0.0000
50	30	0.0140	0.0087	0.0000	0.0000
	40	0.0081	0.0043	0.0000	0.0000
	50	0.0050	0.0022	0.0000	0.0000

Table 3: Failure Rate Average of curtate Weibull Distribution

x	k	$\alpha = 0.5$ $\lambda = 0.5$	$\alpha = 0.5$ $\lambda = 1.2$	$\alpha = 1.2$ $\lambda = 0.5$	$\alpha = 1.2$ $\lambda = 1.2$
20	10	0.0680	0.1033	0.6309	0.9421
	20	0.0629	0.0958	0.6431	0.9472
	30	0.0590	0.0899	0.6533	0.9512
	40	0.0559	0.0852	0.6620	0.9545
	50	0.0533	0.0813	0.6697	0.9572
30	10	0.0578	0.0881	0.6555	0.9525
	20	0.0545	0.0831	0.6646	0.9559
	30	0.0518	0.0791	0.6725	0.9587
	40	0.0496	0.0757	0.6794	0.9610
	50	0.0476	0.0728	0.6857	0.9630
40	10	0.0512	0.0781	0.6738	0.9593
	20	0.0488	0.0746	0.6811	0.9618
	30	0.0468	0.0716	0.6875	0.9639
	40	0.0451	0.0690	0.6932	0.9657
	50	0.0436	0.0667	0.6985	0.9673
50	10	0.0464	0.0710	0.6884	0.9643
	20	0.0446	0.0682	0.6944	0.9662
	30	0.0430	0.0659	0.6998	0.9678
	40	0.0417	0.0638	0.7047	0.9693
	50	0.0405	0.0620	0.7092	0.9706

Table 4: Failure Rate of curtate Pareto Distribution

x	θ	$k = 10$	$k = 20$	$k = 30$	$k = 40$	$k = 50$
20	0.5	0.0163	0.0123	0.0099	0.0082	0.0071
	1.2	0.0386	0.0292	0.0235	0.0196	0.0169
30	0.5	0.0123	0.0099	0.0082	0.0071	0.0062
	1.2	0.0292	0.0235	0.0196	0.0169	0.0148
40	0.5	0.0099	0.0082	0.0071	0.0062	0.0055
	1.2	0.0235	0.0196	0.0169	0.0148	0.0132
50	0.5	0.0082	0.0071	0.0062	0.0055	0.0050
	1.2	0.0196	0.0169	0.0148	0.0132	0.0119

Table 5: Reversed Failure Rate of curtate Pareto Distribution

x	θ	$k=10$	$k=20$	$k=30$	$k=40$	$k=50$
20	0.5	0.0724	0.0296	0.0170	0.0112	0.0081
	1.2	0.0616	0.0225	0.0117	0.0072	0.0048
30	0.5	0.0793	0.0339	0.0199	0.0134	0.0098
	1.2	0.0708	0.0278	0.0151	0.0096	0.0066
40	0.5	0.0835	0.0366	0.0219	0.0149	0.0110
	1.2	0.0765	0.0313	0.0176	0.0114	0.0080
50	0.5	0.0862	0.0386	0.0234	0.0161	0.0120
	1.2	0.0803	0.0339	0.0195	0.0129	0.0091

Table 6: Failure Rate Average of curtate Pareto Distribution

x	θ	$k=10$	$k=20$	$k=30$	$k=40$	$k=50$
20	0.5	0.0197	0.0169	0.0150	0.0135	0.0123
	1.2	0.0467	0.0402	0.0355	0.0321	0.0293
30	0.5	0.0141	0.0126	0.0114	0.0104	0.0097
	1.2	0.0335	0.0299	0.0271	0.0249	0.0231
40	0.5	0.0110	0.0100	0.0092	0.0086	0.0080
	1.2	0.0262	0.0238	0.0220	0.0204	0.0191
50	0.5	0.0090	0.0083	0.0078	0.0073	0.0069
	1.2	0.0215	0.0198	0.0185	0.0174	0.0164

Table 7: Mean Residual life of curtate Pareto Distribution

x	θ	$k=10$	$k=20$	$k=30$	$k=40$	$k=50$
20	0.5	64.3117	66.6748	67.6353	69.4423	69.5009
	1.2	24.7374	31.5418	36.1100	38.8128	39.9057
30	0.5	66.4014	70.4552	73.3231	74.0863	74.6931
	1.2	33.1289	38.1844	41.3945	43.0120	43.2229
40	0.5	68.1639	72.7988	76.4395	78.8635	79.7507
	1.2	40.1321	43.8185	45.9286	46.6463	46.1135
50	0.5	69.6728	74.7822	79.0386	82.2763	84.2673
	1.2	46.1016	48.6755	49.8707	49.8274	50.6576

Table 8: D_1 of curtate Pareto Distribution

x	θ	$k=10$	$k=20$	$k=30$	$k=40$	$k=50$
20	0.5	0.0404	0.0551	0.0610	0.0631	0.0636
	1.2	0.0574	0.0654	0.0629	0.0579	0.0528
30	0.5	0.0246	0.0363	0.0423	0.0454	0.0470
	1.2	0.0404	0.0517	0.0536	0.0520	0.0494
40	0.5	0.0165	0.0256	0.0310	0.0342	0.0362
	1.2	0.0294	0.0406	0.0444	0.0449	0.0439
50	0.5	0.0118	0.0191	0.0237	0.0267	0.0287
	1.2	0.0222	0.0324	0.0368	0.0384	0.0385

Table 9: D_2 of curtate Pareto Distribution

x	θ	$k=10$	$k=20$	$k=30$	$k=40$	$k=50$
20	0.5	37.4979	58.1888	71.2387	80.1196	86.4625
	1.2	13.2778	17.9823	19.8376	20.5058	20.6143
30	0.5	31.4939	51.6756	65.6519	75.8293	83.5028
	1.2	13.9584	20.5692	23.9554	25.7197	26.5931
40	0.5	27.1430	46.1156	60.0700	70.7061	79.0279
	1.2	13.9674	21.7136	26.2518	28.9765	30.6080
50	0.5	23.8560	41.5125	55.0605	65.7367	74.3230
	1.2	13.7122	22.1211	27.4874	30.9888	33.2893