Ageing Properties of Curtate Future Life Time
Kanchan Jain* and Harmanpreet Singh Kapoor
Department of Statistics, Panjab University, Chandigarh- 160014, India
Corresponding Author (Email: jaink14@gmail.com)

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ABSTRACT
In insurance sector, for determining the premium of an insured aged $x$, the insurer is interested not only in the complete future lifetime $T(x)$, but also in the individual’s curtate future lifetime $K(x) = [T(x)]$. In this paper, we derive expressions for the reliability measures of $K(x)$ and explore some of its ageing properties.

Keywords: Complete Future Lifetime, Curtate Future Lifetime, IFR, IFRA, DMRL, NBU, NBUE, HNBUE.

1. Introduction
For a new-born child, the age at death is a continuous random variable $X$ with cumulative distribution function (cdf) $F(x)$ and survival function $\bar{F}(x) = P(X > x)$ for $x > 0$. For a non-living object, $X$ represents the age at failure. In actuarial science, $\bar{F}(x)$ is the survival probability that a person/object survives for at least $x$ years and is denoted by $\bar{P}_x$. For $X$ with support on $\{1,2,\ldots\}$, $\bar{P}_x = P(X \geq x)$ and for support $\{0,1,2,\ldots\}$, $\bar{P}_x = P(X > x)$.

The notation $(x)$ is used to denote a life aged $x$ and $T(x)$ denotes the complete future lifetime of $(x)$. According to International Actuarial Notations given in 1949, $t q_x$ is the cdf of $T(x)$ at $t$ and gives the probability that $(x)$ dies within $t$ years. The survival function of $T(x)$ is

$$t p_x = 1 - t q_x = P[T(x) > t] = P[X > x + t | X > x] = \frac{\bar{P}_{x+t}}{\bar{P}_x}. \quad (1)$$

The probability that $(x)$ will die between ages $x + t$ and $x + t + u$ is written as

$$t u q_x = t p_x - t p_{x+t} = t p_x - t p_x u p_{x+t} = t p_x u q_{x+t} = \frac{\bar{P}_{x+t} - \bar{P}_{x+t+u}}{\bar{P}_x}. \quad (1)$$
For \( u = 1 \), \( t_{x+1} = t_x \) and \( 0 = q_x \). The curtate future lifetime of \( (x) \) is the discrete number of future years completed by \( (x) \) prior to death and is written as
\[
K(x) = \lfloor T(x) \rfloor : \text{the greatest integer less than or equal to } T(x).
\]

\( K(x) \) is a discrete random variable whereas \( T(x) \) is a continuous random variable. One can refer to Neill (1977), Gerber (1990), Bowers et al. (1997), Slud (2001), Borowiak (2003), Zhu (2007) and Dickson et al. (2009) for the above notations and definitions.

In lifetime analysis, an important aspect is to find a lifetime distribution that can adequately describe the ageing behaviour of the concerned life. Lifetimes are continuous in nature and hence many continuous life distributions have been proposed in the literature. On the other hand, discrete failure data arise in several common situations. For example, reports on insured’s deaths are collected half yearly or annually and the observations are the number of deaths without specification of the occurrence of events. Sometimes, it is essential to count the complete number of years for which a patient has survived after going through a severe operation. The curtate future lifetime plays a significant role in assurance contracts and discrete life annuities when the benefit is payable at the end of year of death of the claimant/insured. As compared to continuous failure data, interest in discrete analogue arose relatively late. It was only briefly mentioned by Barlow and Proschan (1981). For earlier works on discrete lifetime distributions, one can see Salvia and Bollinger (1982), Xekalaki (1983), Padgett and Spurrier (1985) and Ebrahimi (1986).

Many authors have studied various ageing properties of complete future life time \( T(x) \) (termed as residual life in reliability theory) (Ref. Deshpande et al. (1986), Gupta (1987), Lai and Xie (2006) and Nanda et al. (2010)). But the literature is devoid of study of the ageing properties of curtate future lifetime. Hence, we are motivated to explore the ageing properties of \( K(x) \), the curtate future lifetime in terms of ageing properties of \( X \). For this purpose, the underlying distribution of \( X \) is assumed to be Exponential, Weibull, Pareto and Burr.

As we are interested in finding the reliability measures and exploring the ageing properties of \( K(x) \), the relevant definitions are listed below (Ref. Johnson, Kemp and Kotz (2005), Lai and Xie (2006) and Sudheesh and Dewan (2009)).
For $\tilde{P}_l = P[Y > l]$ and $P_l = P[Y \leq l],$

1. the discrete failure rate (FR) is the amount of risk associated with an item at time $l$ and is defined as

$$h_y(l) = \frac{P[Y = l]}{P[Y \geq l]} = \frac{\tilde{P}_l - \tilde{P}_{l+1}}{\tilde{P}_l}, \ l = 0, 1, 2 \ldots \quad (2)$$

2. the discrete reversed failure rate (RFR) is defined as:

$$\tau_y(l) = \frac{P[Y = l]}{P[Y \leq l]} = \frac{\tilde{P}_l - \tilde{P}_{l+1}}{P_l}, \ l = 0, 1, 2 \ldots \quad (3)$$

3. the discrete failure rate average (FRA) is

$$\frac{1}{(l+1)} \sum_{j=0}^{l} h_y(j) = \frac{1}{(l+1)} \sum_{j=0}^{l} \frac{P[Y = j]}{P[Y \geq j]}$$

$$= \frac{1}{(l+1)} \sum_{j=0}^{l} \frac{\tilde{P}_j - \tilde{P}_{j+1}}{\tilde{P}_j}, \ l = 0, 1, 2 \ldots \quad (4)$$

4. the discrete mean residual life (MRL) is defined as

$$m_y(l) = \sum_{j=l}^{\infty} \frac{P[Y \geq j]}{P[Y \geq l]} = \sum_{j=l}^{\infty} \frac{\tilde{P}_j}{\tilde{P}_l}, \ l = 0, 1, 2 \ldots \quad (5)$$

Some of the discrete ageing properties of $Y$ given by Kelfsjo (1982) and Johnson, Kemp and Kotz (2005) are reproduced below:

1. **Discrete IFR (DFR)**: A discrete distribution with infinite support has a monotonically increasing (decreasing) failure rate with time according as

$$\frac{P_{k+1}}{P_k} > (<) \frac{P_{k+2}}{P_{k+1}} \quad (6)$$

where $p_k = P[Y = k], \ k = 0, 1, 2 \ldots$
2. **Discrete IFRA (DFRA):** A discrete lifetime distribution has an increasing or decreasing failure rate average according as

\[
\frac{\sum_{j=0}^{k} h_y(j)}{k+1} > (<) \frac{\sum_{j=0}^{k-1} h_y(j)}{k}, \quad k = 1, 2, \ldots \tag{7}
\]

or

\[
\frac{\sum_{j=0}^{k+1} h_y(j)}{k+2} > (<) \frac{\sum_{j=0}^{k} h_y(j)}{k+1}, \quad k = 0, 1, 2, \ldots
\]

3. **Discrete IMRL (DMRL):** An increasing (decreasing) mean residual life is determined by

\[
\sum_{j=k}^{\infty} \left( \frac{\bar{P}_j}{\bar{P}_k} - \frac{\bar{P}_{j+1}}{\bar{P}_{k+1}} \right) > (<) 0, \quad k = 0, 1, 2, \ldots \tag{8}
\]

4. **Discrete NBU (NWU):** - The distribution of \( Y \) is new better (worse) than used if

\[
\bar{P}_{j+k} < (> \bar{P}_j \bar{P}_k \quad \text{for } j, k = 0, 1, 2, \ldots \tag{9}
\]

5. **Discrete NBUE (NWUE):** \( Y \) has a distribution which is new better (worse) than used in expectation according as:

\[
\sum_{j=0}^{\infty} \bar{P}_{j+k} < (>) \bar{P}_k \sum_{j=0}^{\infty} \bar{P}_j, \quad k = 0, 1, 2, \ldots \tag{10}
\]

6. **Discrete HNBU (HNBUE):** \( Y \) is harmonically new better (worse) than used in expectation if

\[
\sum_{j=k}^{\infty} \bar{P}_j < (> \mu \left(1 - \frac{1}{\mu} \right)^k, k = 0, 1, 2, \ldots \tag{11}
\]

where \( \mu = \sum_{j=0}^{\infty} P[Y > j] = \sum_{j=0}^{\infty} \bar{P}_j. \)

The paper is organised as follows:

In Section 2, the expressions for reliability measures of \( K(x) \) are derived. The conditions for holding of ageing properties of \( K(x) \) have been presented in
Section 3. Section 4 investigates the ageing behaviour of \( K(x) \) under the assumption that \( X \) follows Exponential, Weibull, Pareto or Burr distribution.

2. Reliability Measures of \( K(x) \)

In this section, we find the expressions for different reliability measures of \( K(x) \). The probability function of \( K(x) \) is given by

\[
P[K(x) = k] = k p_x q_{x+k} = k |q_x \quad (\text{Ref. Bowers et al. (1997)}).
\]

1. The failure rate (FR) function of \( K(x) \) is given by

\[
h_{K(x)}(k) = \frac{P[K(x) = k]}{P[K(x) \geq k]} = \frac{k |q_x}{\sum_{l=k}^{\infty} l |q_x} \quad k = 0, 1, 2 \ldots
\]

Using (1), \( h_{K(x)}(k) = \frac{\bar{P}_{x+k} - \bar{P}_{x+k+1}}{\bar{P}_{x+k}} = q_{x+k} \).

2. The reversed failure rate (RFR) of \( K(x) \) can be written as

\[
\tau_{K(x)}(k) = \frac{P[K(x) = k]}{P[K(x) \leq k]} = \frac{k |q_x}{\sum_{l=0}^{k} l |q_x}
\]

\[
= \frac{\bar{P}_{x+k} - \bar{P}_{x+k+1}}{\bar{P}_{x} - \bar{P}_{x+k+1}} = q_{x+k} \left( \frac{\bar{P}_{x+k}}{\bar{P}_{x} - \bar{P}_{x+k+1}} \right)
\]

\[
= q_{x+k} \left( \frac{k p_x}{1 - \frac{1}{k+1} p_x} \right), k = 0, 1, 2 \ldots
\]

3. The failure rate average (FRA) of \( K(x) \) is given by

\[
\frac{1}{k+1} \sum_{j=0}^{k} \frac{P[K(x) = j]}{P[K(x) \geq j]} = \frac{1}{k+1} \sum_{j=0}^{k} \frac{j |q_x}{\sum_{l=j}^{\infty} l |q_x}
\]

\[
= \frac{1}{k+1} \sum_{j=0}^{k} q_{x+j}.
\]
4. The mean residual life (MRL) of $K(x)$ is

$$m_{K(x)}(k) = \sum_{j=k+1}^{\infty} P[K(x) \geq j] = \sum_{j=k+1}^{\infty} \frac{\sum_{l=j}^{\infty} q_{x+l}}{\sum_{l=k}^{\infty} q_{x+l}} = \sum_{j=k+1}^{\infty} \frac{\sum_{l=j}^{\infty} (\bar{P}_{x+l} - \bar{P}_{x+l+1})}{\sum_{l=k}^{\infty} (\bar{P}_{x+l} - \bar{P}_{x+l+1})} = \sum_{j=k+1}^{\infty} \frac{\bar{P}_{x+j}}{\bar{P}_{x+k}} = \sum_{j=1}^{\infty} jP_{x+k}.$$ 

For $k = 0$, $m_{K(x)}(0) = E[K(x)] = \hat{e}_x$ : Curtate expectation of life (Ref. Bowers et al. (1997))
The ageing properties of $K(x)$ are explored in the next section.

3. Ageing properties of $K(x)$

The following theorem specifies the conditions required for possessing of various ageing properties by $K(x)$.

**Theorem 1**: The distribution of curtate future lifetime $K(x)$ is

1. IFR (DFR) if $(k+1)p_x - k+2p_x)^2 > (k)p_x - k+1p_x) (k+2p_x - k+3p_x)$, $k = 0, 1, 2, \ldots$;

2. IFRA (DFRA) if $\left(\sum_{j=0}^{k-1} p_{x+j} - k p_{x+k}\right) > (k) 0$;

3. DMRL (IMRL) if $\sum_{j=k+1}^{\infty} \frac{jP_x}{kP_x} - \sum_{j=k+2}^{\infty} \frac{jP_x}{k+1P_x} > (k) 0$;

4. NBU (NWU) if $j+kP_x < (>) jP_x kP_x$;

5. NBUE (NWUE) if $\sum_{j=0}^{\infty} j+kP_x < (>) kP_x \sum_{j=0}^{\infty} jP_x$;

6. HNBUE (HNWUE) if $\sum_{j=k}^{\infty} jP_x < (>) \mu (1 - \frac{1}{\mu})^{-k}$ where $\mu = \sum_{j=1}^{\infty} jP_x$.

**Proof**: See the appendix

In the sequel, the distributions of $K(x)$ will be designated as Curtate Distributions.

**Remark 1**: For exploring the ageing properties NBU (NWU), NBUE (NWUE) and HNBUE (HNWUE) of curtate distributions, we consider the following differences
1. \( D_1 = \frac{\bar{P}_{x+j+k}}{\bar{P}_x} - \frac{\bar{P}_{x+j}}{\bar{P}_x} \frac{\bar{P}_{x+k}}{\bar{P}_x} \);

2. \( D_2 = \sum_{j=0}^{\infty} \frac{\bar{P}_{x+j+k}}{\bar{P}_x} - \frac{\bar{P}_{x+k}}{\bar{P}_x} \sum_{j=0}^{\infty} \frac{\bar{P}_{x+j}}{\bar{P}_x} \);

3. \( D_3 = \sum_{j=k}^{\infty} \frac{\bar{P}_{x+j}}{\bar{P}_x} - \mu \left( 1 - \frac{1}{\mu} \right)^k \).

The conclusions are based on positive or negative signs of the above differences. This is proved either mathematically or through figures in case the mathematical form is not tractable.

In the next section, we explore the ageing properties of \( K(x) \) when \( X \), the new born’s age at death follows some pre-assumed distribution.

4. Ageing Properties

The ageing properties of \( K(x) \) are investigated when \( X \), the new born’s age at death follows Exponential, Weibull, Pareto or Burr distribution.

4.1. Curtate Exponential Distribution

Let \( X \sim \text{Exp}(\lambda) \), \( \lambda > 0 \), then

\[ P_x = e^{-\lambda x} \quad \text{and} \quad kP_x = \frac{\bar{P}_{x+k}}{\bar{P}_x} = e^{-\lambda k}. \]

Hence the probability mass function (pmf) of \( K(x) \) is

\[ P[K(x) = k] = \frac{\bar{P}_{x+k} - \bar{P}_{x+k+1}}{\bar{P}_x} = e^{-\lambda k} (1 - e^{-\lambda}) \text{ for } \lambda > 0 \text{ and } k = 0, 1, 2, \ldots \]

**Theorem 2**: If \( X \sim \text{Exp}(\lambda) \), then the distribution of curtate future lifetime \( K(x) \) is (i) IFR and DFR (ii) DRFR (iii) IFRA and DFRA (iv) DMRL and IMRL (v) NBU and NWU (vi) NBUE and NWUE (vii) HNBUE and HNWUE.

**Proof**: See the appendix.
4.2. Curtate Weibull Distribution

Let \( X \sim \text{Weibull}(\alpha, \lambda) \), then \( \bar{P}_x = e^{-(\lambda x)^\alpha} \) for \( \lambda > 0, \alpha > 0, x > 0 \) gives

\[
kp_x = e^{(\lambda x)^\alpha} e^{-\left[\lambda(x+k)\right]^\alpha}.
\]

The pmf of \( K(x) \) is given by

\[
P[K(x) = k] = e^{-\lambda^\alpha \left[(x+k)^\alpha - (x)^\alpha\right]} - e^{-\lambda^\alpha \left[(x+k+1)^\alpha - (x)^\alpha\right]}
= e^{(\lambda x)^\alpha} \left[e^{-\left[\lambda(x+k)\right]^\alpha} - e^{-\left[\lambda(x+k+1)\right]^\alpha}\right]. \tag{12}
\]

From (2), (3), (4) and (5) FR, RFR, FRA and MRL for Curtate Weibull distribution are derived as:

\[
\text{FR} = 1 - e^{-\lambda^\alpha \left[(x+k+1)^\alpha - (x+k)^\alpha\right]},
\]

\[
\text{RFR} = \frac{e^{-\lambda^\alpha \left[(x+k)^\alpha - (x)^\alpha\right]} - e^{-\lambda^\alpha \left[(x+k+1)^\alpha - (x)^\alpha\right]}}{1 - e^{-\lambda^\alpha \left[(x+k+1)^\alpha - (x)^\alpha\right]}},
\]

\[
\text{FRA} = \frac{1}{k+1} \sum_{j=0}^{k} \left(1 - e^{-\lambda^\alpha \left[(x+j+1)^\alpha - (x+j)^\alpha\right]}\right)
\approx 1 - \sum_{j=0}^{k} e^{-\lambda^\alpha \left[(x+j)^\alpha - (x)^\alpha\right]} - \alpha (\alpha - 1) \lambda^\alpha (x+j)^{\alpha-2} \frac{\alpha^2}{2}
\]
when \( \alpha \) is an integer.

\[
\text{MRL} = \sum_{j=k}^{\infty} e^{-\lambda^\alpha \left[(x+j+1)^\alpha - (x)^\alpha\right]}
\]

Theorem 3: If \( X \sim \text{Weibull}(\alpha, \lambda) \), then the distribution of curtate future lifetime \( K(x) \) is

(i) IFR (DFR) for \( \alpha > (\alpha <) 1 \) and any \( \lambda \);
(ii) DRFR if

(a) \( \alpha > 1 \) and \( \lambda > \left(\frac{\alpha - 1}{\alpha (x+k)^\alpha}\right)^\frac{1}{\alpha} \);
(b) \( \alpha < 1 \) and all \( \lambda > 0 \).
(iii) IFRA (DFRA) for \( \alpha > (\alpha < 1) \) and all \( \lambda > 0 \);

Proof See the appendix.

Tables 1 – 3 (given in the appendix) display the values of failure rate, reversed failure rate and failure rate average for different values of \( \alpha \) and \( \lambda \).
and for \( x = 20, 30, 40, 50 \). The tables are based on hypothetical data which can arise in real life situations.

On the basis of Table 1, it can be concluded that

- for each \( \lambda > 0 \), and
  (i) \( 0 < \alpha < 1 \), FR decreases in \( k \) as well as \( x \).
  (ii) \( \alpha > 1 \), FR increases in \( k \) as well as \( x \).

- RFR
  (i) decreases if \( 0 < \alpha < 1 \) and \( \lambda > 0 \) and
  (ii) increases if \( \alpha > 1 \) and \( \lambda > \left( \frac{\alpha - 1}{\alpha(x+k)^{\alpha}} \right) \frac{1}{\alpha} \).

- similarly for \( \lambda > 0 \), FRA decreases in \( k \) and \( x \) if \( 0 < \alpha < 1 \) and increases in \( k \) as well as \( x \) if \( \alpha > 1 \).

The mean residual life function is plotted in Figure 1a – 1b.

Figures 2 – 4 plot the following differences

\[
D_1 = e^{-\lambda \alpha (x+j+k)^{\alpha}-x^{\alpha}} - e^{-\lambda \alpha (x+j)^{\alpha}+(x+k)^{\alpha}-2x^{\alpha}};
\]

\[
D_2 = \sum_{j=0}^{\infty} e^{-\lambda \alpha (x+j+k)^{\alpha}-x^{\alpha}} - e^{-\lambda \alpha (x+k)^{\alpha}-x^{\alpha}} \sum_{j=0}^{\infty} e^{-\lambda \alpha (x+j)^{\alpha}-x^{\alpha}};
\]

and

\[
D_3 = \sum_{j=k}^{\infty} e^{-\lambda \alpha (x+j)^{\alpha}-x^{\alpha}} - \sum_{j=1}^{\infty} e^{-\lambda \alpha (x+j)^{\alpha}-x^{\alpha}} \left( 1 - \frac{1}{\sum_{j=1}^{\infty} e^{-\lambda \alpha (x+j)^{\alpha}-x^{\alpha}}} \right)^k.
\]

From Figures 1a – 4, it can be concluded that for any \( \lambda > 0 \), \( K(x) \) is

- DMRL (IMRL) for \( \alpha > (\leq) 1 \);
- NBU (NWU) for all \( \alpha > (\leq) 1 \);
- NBUE (NWUE) for all \( \alpha > (\leq) 1 \);
- HNBU (HNWUE) for \( \alpha > (\leq) 1 \).

**Remark (2):** Exponential Distribution is a special case of Weibull distribution for \( \alpha = 1 \), and in that case both the distributions exhibit the same ageing behaviour.
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Figure 1a: MRL for $\alpha < 1$

Figure 1b: MRL for $\alpha > 1$

Figure 2: $D_1$ for Curtate Weibull Distri.

Figure 3: $D_2$ for Curtate Weibull Distri.

Figure 4: $D_3$ for Curtate Weibull Distribution

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4.3. Curtate Pareto distribution

If $X \sim \text{Pareto} (\theta)$, then $\hat{P}_x = \frac{1}{x^\theta}$, $\theta > 0$, $x > 0$.

This gives $k \hat{p}_x = \left( \frac{x}{x+k} \right)^\theta$, $x > 0$. Hence,

$$P[K(x) = k] = \left( \frac{x}{x+k} \right)^\theta - \left( \frac{x}{x+k+1} \right)^\theta = \left( 1 + \frac{k}{x} \right)^{-\theta} - \left( 1 + \frac{k+1}{x} \right)^{-\theta} \text{ for } k = 0, 1, 2, \ldots$$

Using (2) - (5), FR, RFR, FRA and MRL for Curtate Pareto distribution are derived as:

- $\text{FR} = 1 - \frac{(x+k)^\theta}{(x+k+1)^\theta}$
- $\text{RFR} = \left( \frac{(x+k+1)^\theta - (x+k)^\theta}{(x+k)^\theta - (x)^\theta} \right) \left( \frac{x^\theta}{(x+k+1)^\theta} \right)$
- $\text{FRA} = \frac{1}{k+1} \sum_{j=0}^{k} \left( 1 - \frac{(x+j)^\theta}{(x+j+1)^\theta} \right)$
- $\approx -k - \theta \left( \psi(0,x) - \psi(0,1+k+x)(1+\theta) (\psi(1,x) - \psi(1,1+k+x)) \right)$, when $\theta$ is an integer and $\psi(n,x)$, the digamma function denotes the $n^{th}$ derivative of gamma function.
- $\text{MRL} = \sum_{j=k+1}^{\infty} \frac{(k+x)^\theta}{(j+x)^\theta}$ for $\theta$, an integer.

**Theorem 4:** If $X \sim \text{Pareto}(\theta)$, then the distribution of curtate future lifetime $K(x)$ is (i) DFR (ii) DRFR (iii) DFRA (iv) IMRL (v) NWU (vi) NWUE.

**Proof** See the appendix.

For concluding about HNWUE, we consider

$$D_3 = \sum_{j=k}^{\infty} \left( \frac{x}{j+x} \right)^\theta - \sum_{j=1}^{\infty} \left( \frac{x}{j+x} \right)^\theta \left( 1 - \frac{1}{\sum_{j=1}^{\infty} \left( \frac{x}{j+x} \right)^\theta} \right) \theta$$
As it is difficult to decide the sign of $D_3$, we try to conclude by plotting $D_3$ in Figure 5.

Since $D_3 > 0$ for all $\theta$, hence the Curtate Pareto distribution is HNWUE.

The values of FR, RFR, FRA and MRL have been displayed in Tables 4 – 7 for curtate Pareto distribution. From these tables, we can conclude that

- the values of FR decrease as initial age $x$ and $k$ increase for all $\theta > 0$.
- the RFR is a decreasing function of $x$ and $k$.
- FRA values show a decreasing trend as the values of $x$ and $k$ increase for all values of $\theta$.
- MRL is an increasing function of initial age $x$ and $k$ for all $\theta$.

4.4. Curtate Burr Distribution

If $X \sim \text{Burr}(\theta, c)$, then $\bar{P}_x = \frac{1}{(1 + x^c)^\theta}$, $\theta > 0, c > 0$ and $x > 0$. 
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This gives

\[ k p_x = \left( \frac{1 + x^c}{1 + (x + k)^c} \right)^\theta. \]

Hence

\[ P[K(x) = k] = \left( \frac{1 + x^c}{1 + (x + k)^c} \right)^\theta - \left( \frac{1 + x^c}{1 + (x + k + 1)^c} \right)^\theta = (1 + x^c)^\theta \left( \frac{1}{(1 + (x + k)^c)^\theta} - \frac{1}{(1 + (x + k + 1)^c)^\theta} \right). \]

Using (2) - (5), the FR, RFR, FRA and MRL for Curtate Burr distribution are

\[ \text{FR} = 1 - \frac{(1 + (x + k)^c)^\theta}{(1 + (x + k + 1)^c)^\theta}, \]

\[ \text{RFR} = \left( \frac{(1 + (x + k + 1)^c)^\theta - (1 + (x + k)^c)^\theta}{(1 + (x + k + 1)^c)^\theta - (1 + x^c)^\theta} \right) \left( \frac{(1 + x^c)^\theta}{(1 + (x + k)^c)^\theta} \right), \]

\[ \text{FRA} = \frac{1}{k + 1} \sum_{j=0}^{k} \left( 1 - \frac{(1 + (x + j)^c)^\theta}{(1 + (x + j + 1)^c)^\theta} \right) \approx 1 - \frac{1}{k + 1} \sum_{j=0}^{k} \left( 1 + \frac{c(x + j)^{c-1} + c(c - 1)(x + j)^{c-2}}{(1 + (x + j)^c)} \right)^{-\theta}, \]

\[ \text{MRL} = \sum_{j=k+1}^{\infty} \frac{(1 + (x + j)^c)^\theta}{(1 + (x + j)^c)^\theta} \approx \sum_{j=1}^{\infty} \frac{1}{(1 + j c(x + k)^{c-1} + \frac{j c (c - 1) (x + k)^{c-2}}{2})} \]

where \( \theta \) and \( c \) are integers.

For Curtate Burr distribution, we conclude about the ageing properties by looking at the plots of different reliability measures as shown in Figures 6 - 9. Figures 10 - 12 plot the differences

\[ D_1 = \left( \frac{1 + x^c}{1 + (x + j + k)^c} \right)^\theta - \left( \frac{1 + x^c}{1 + (x + j)^c} \right)^\theta \left( \frac{1 + x^c}{1 + (x + k)^c} \right)^\theta; \]

\[ D_2 = \sum_{j=0}^{\infty} \left( \frac{1 + x^c}{1 + (x + j + k)^c} \right)^\theta - \left( \frac{1 + x^c}{1 + (x + j)^c} \right)^\theta \sum_{j=0}^{\infty} \left( \frac{1 + x^c}{1 + (x + j)^c} \right)^\theta; \]
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Figure 6: Failure Rate

Figure 7: Reversed failure Rate

Figure 8: Failure Rate Average

Figure 9: Mean Residual life

Figure 10: $D_1$ for Curtate Burr Distri.

Figure 11: $D_2$ for Curtate Burr Distri.
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\[ D_3 = \sum_{j=k}^{\infty} \left( \frac{1 + x^c}{1 + (x+j)^c} \right)^\theta - \sum_{j=1}^{\infty} \left( \frac{1 + x^c}{1 + (x+j)^c} \right)^\theta \left( 1 - \frac{1}{\sum_{j=1}^{\infty} \left( \frac{1 + x^c}{1 + (x+j)^c} \right)^\theta} \right). \]

Figures 6 – 12 help us to conclude that \( K(x) \) is DFR, DRFR, DFRA and IMRL for different parametric combinations. It can also be concluded that for all \( \theta \) and \( c \), \( K(x) \) is NWU, NWUE and HNWUE.

On the basis of the above discussion, it is concluded that the considered ageing properties of the distribution of \( X \) are preserved by the curtate future lifetime distributions.

5. Conclusions

In this paper, we have studied the ageing properties of the curtate future lifetime distributions. It is shown that the ageing properties viz IFR (DFR), IFRA (DFRA), DMRL(IMRL), NBU (NWU), NBUE (NWUE) and HNBUE (HNWUE) are preserved by the curtate future lifetime distributions, when the new born’s age at death follows Exponential, Weibull, Pareto and Burr Distributions.
Acknowledgment

The first author is grateful to University Grants Commission, Government of India, for providing financial support for this work.

References


17. Slud, E. V. (2001) — Actuarial Mathematics and Life Table Statistics. Lecture notes, Mathematics Department, University of Maryland, USA.


Appendix

Proof of Theorem 1:
(i) Using (6), $K(x)$ will have an increasing (decreasing) failure rate if

$$\frac{P[K(x) = k + 1]}{P[K(x) = k]} > (<) \frac{P[K(x) = k + 2]}{P[K(x) = k + 1]} \quad k = 0, 1, 2, 3, \ldots$$

that is, if

$$\frac{\tilde{p}_{x+k+1} - \tilde{p}_{x+k+2}}{\tilde{p}_{x+k} - \tilde{p}_{x+k+1}} > (<) \frac{\tilde{p}_{x+k+2} - \tilde{p}_{x+k+3}}{\tilde{p}_{x+k+1} - \tilde{p}_{x+k+2}}$$

This gives

$$(\tilde{p}_{x+k+1})^2 + (\tilde{p}_{x+k+2})^2 - \tilde{p}_{x+k+1} \tilde{p}_{x+k+2} - \tilde{p}_{x+k} \tilde{p}_{x+k+2} + \tilde{p}_{x+k} \tilde{p}_{x+k+3} - \tilde{p}_{x+k+1} \tilde{p}_{x+k+3} > (<) 0.$$

Alternatively, $K(x)$ has IFR (DFR) if

$$(k+1p_x - k+2p_x)^2 > (<) (k+1p_x)(k+2p_x - k+3p_x), k = 0, 1, 2, 3, \ldots$$

Using, $k+1p_x = kp_x \quad 1p_{x+k}$ and $k+2p_x = kp_x \quad 2p_{x+k}$, this is equivalent to

$$(1p_{x+k} - 2p_{x+k})^2 > (<) (1-1p_{x+k})(2p_{x+k} - 3p_{x+k}).$$

This gives

$$\frac{1p_{x+k} - 2p_{x+k}}{1-1p_{x+k}} > (<) \frac{2p_{x+k} - 3p_{x+k}}{1p_{x+k} - 2p_{x+k}},$$

that is

$$\frac{jp_{x+k} - j+1p_{x+k}}{j-1p_{x+k} - jp_{x+k}}$$

is non - increasing (non - decreasing) in $j$.

(ii) $K(x)$ will have an increasing or decreasing failure rate average (IFRA or DFRA) if

$$\frac{\sum_{j=0}^{k} h_{K(x)}(j)}{k + 1}$$

is increasing (decreasing) in $k$.

This happens if
Ageing Properties of Curtate Future Life Time

\[ \sum_{j=0}^{k} h_{K(x)}(j) \frac{1}{k+1} > \left( \sum_{j=0}^{k-1} h_{K(x)}(j) \right), \quad k = 0, 1, 2, 3 \ldots \]

that is, if

\[ \sum_{j=0}^{k} \left( \bar{P}_{x+j} - \bar{P}_{x+j+1} \right) (k+1) \bar{P}_{x+j} > \left( \sum_{j=0}^{k-1} \left( \bar{P}_{x+j} - \bar{P}_{x+j+1} \right) \right) (k) \bar{P}_{x+j}. \]

This means that

\[ \left( \sum_{j=0}^{k-1} \frac{\bar{P}_{x+j+1}}{P_{x+j}} - k \frac{\bar{P}_{x+k+1}}{P_{x+k}} \right) > (>) 0. \]

Since \( k p_x = \frac{\bar{P}_{x+k}}{P_x} \), the above inequality can be expressed as

\[ \left( \sum_{j=0}^{k-1} p_{x+j} - k p_{x+k} \right) > (>) 0 \]

or

\[ \left( \sum_{j=0}^{k} p_{x+j} - (k+1) p_{x+k+1} \right) > (>) 0. \]

It means that the sum of probabilities of survival up to \((x+k)\)th year is greater or less than \(k+1\) times the survival probability of life aged \((x+k+1)\) during the next one year.

(iii) Decreasing (Increasing) mean residual life DMRL (IMRL) of \( K(x) \) is indicated if

\[ \sum_{j=k+1}^{\infty} \left( \frac{P[K(x) \geq j]}{P[K(x) \geq k]} - \frac{P[K(x) \geq j+1]}{P[K(x) \geq k+1]} \right) < (>) 0. \]

This holds if

\[ \sum_{j=k+1}^{\infty} \frac{\bar{P}_{x+j}}{P_{x+k}} - \sum_{j=k+2}^{\infty} \frac{\bar{P}_{x+j}}{P_{x+k+1}} < (>) 0. \]

Alternatively,

\[ \sum_{j=k+1}^{\infty} \frac{jP_x}{kP_x} - \sum_{j=k+2}^{\infty} \frac{jP_x}{k+1P_x} < (>) 0, \]
that is

\[ \sum_{j=k+1}^{\infty} \frac{jP_x}{kP_x} \]

is non-decreasing or non-increasing in \( j > 0 \).

(iv) The distribution of \( K(x) \) will be new better (worse) than used (NBU (NWU)) if the following condition is satisfied

\[ P[K(x) = j + k] < (>) P[K(x) = j]P[K(x) = k] \]

that is

\[ \frac{\tilde{P}_{x+j+k}}{\tilde{P}_x} < (>) \frac{\tilde{P}_{x+j}}{\tilde{P}_x} \frac{\tilde{P}_{x+k}}{\tilde{P}_x} \]

which happens if

\[ j+kP_x < (>) jP_x kP_x \quad \text{for } j, k = 0, 1, 2, 3, \ldots \]

This means that \( K(x) \) is NBU (NWU) if the probability that \( (x) \) survives for \( (j+k) \) years is less (greater) than the product of survival probabilities of \( (x) \) till \( j \) and \( k \) years.

(v) The distribution of \( K(x) \) will have a new better (worse) than used in expectation (NBUE (NWUE)) if the following condition is satisfied

\[ \sum_{j=0}^{\infty} P[K(x) \geq j + k] < (>) \sum_{j=0}^{\infty} P[K(x) \geq j] \sum_{j=0}^{\infty} P[K(x) \geq j] \]

that is,

\[ \sum_{j=0}^{\infty} \frac{\tilde{P}_{x+j+k}}{\tilde{P}_x} < (>) \frac{\tilde{P}_{x+k}}{\tilde{P}_x} \sum_{j=0}^{\infty} \frac{\tilde{P}_{x+j}}{\tilde{P}_x} \]

or

\[ \sum_{j=0}^{\infty} \frac{\tilde{P}_{x+j+k}}{\tilde{P}_x} < (>) \frac{\tilde{P}_{x+k}}{\tilde{P}_x} \sum_{j=0}^{\infty} \frac{\tilde{P}_{x+j}}{\tilde{P}_x}, \quad \text{using } k+jP_x = kP_x jP_{x+k}. \]

(vi) The distribution of \( K(x) \) will be Harmonically new better (worse) than used in expectation (HNBUE (HNWUE)) if

\[ \sum_{j=k}^{\infty} P[K(x) \geq j] < (>) \mu \left( 1 - \frac{1}{\mu} \right)^k, \]
that is,
\[ \sum_{j=k}^{\infty} \frac{\bar{P}_{x+j}}{P_x} < (>) \mu \left( 1 - \frac{1}{\mu} \right) \]
or
\[ \sum_{j=k}^{\infty} jP_x < (>) \mu \left( 1 - \frac{1}{\mu} \right) \]
where \( \mu = \sum_{j=1}^{\infty} \frac{\bar{P}_{x+j}}{P_x} = \sum_{j=1}^{\infty} jP_x \).

**Proof of Theorem 2**
Using (2) - (5) the FR, RFR, FRA and MRL for Curtate Exponential distribution are
\[ (1 - e^{-\lambda}), \frac{e^{-\lambda k} (1 - e^{-\lambda})}{(1 - e^{-\lambda (k+1)})}, (1 - e^{-\lambda}), \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \]
respectively.

(i), (iii) and (iv) follow since FR, FRA and MRL are independent of \( k \).

(ii) follows since
\[ \frac{d}{dk} \left( \frac{e^{-\lambda k} (1 - e^{-\lambda})}{(1 - e^{-\lambda (k+1)})} \right) = \frac{-\lambda e^{-\lambda k}}{(1 - e^{-\lambda (k+1)})^2} < 0. \]

(v) Since
\[ \frac{\bar{P}_{x+j+k}}{\bar{P}_x} = e^{-\lambda (j+k)} = \frac{\bar{P}_{x+j+k}}{\bar{P}_x} \frac{\bar{P}_{x+k}}{\bar{P}_x}, \]
hence Curtate Exponential is both NBU and NWU.

(vi) The curtate exponential distribution is both NBUE and NWUE. Since
\[ \sum_{j=0}^{\infty} \frac{\bar{P}_{x+j+k}}{P_x} = \frac{e^{-\lambda k}}{1 - e^{-\lambda}} = \frac{\bar{P}_{x+k}}{\bar{P}_x} \sum_{j=0}^{\infty} \frac{\bar{P}_{x+j}}{P_x}. \]

(vii) The ageing properties HNBUE and HNWUE follow since
\[ \sum_{j=k}^{\infty} \frac{\bar{P}_{x+j}}{P_x} = \frac{e^{-\lambda k}}{1 - e^{-\lambda}} = \mu \left( 1 - \frac{1}{\mu} \right), \quad k = 0, 1, 2, 3 \ldots \]
Proof of Theorem 3:

Let

\[ f(k) = e^{-\lambda^k \{(x+k+1)^\alpha - (x+k)^\alpha\}}. \]

This gives

\[
\frac{df(k)}{dk} = -\alpha \lambda \{ (x+k+1)^\alpha - (x+k)^\alpha \} e^{-\lambda^k \{(x+k+1)^\alpha - (x+k)^\alpha\}}
\]

< (>) 0 for \( \alpha > (<) 1. \)

Hence \( f(k) \) is decreasing for \( \alpha > 1 \) and increasing for \( \alpha < 1. \)

(i) Since \( FR = 1 - f(k) \), hence \( FR \) is increasing for \( \alpha > 1 \) and decreasing for \( \alpha < 1. \)

(ii) Consider \( \frac{d(RFR)}{dk} = \frac{d}{dk} \left[ \frac{e^{-\lambda^k \{(x+k+1)^\alpha - (x+k)^\alpha\}} - e^{-\lambda^k \{(x+k)^\alpha - x^\alpha\}}}{1 - e^{-\lambda^k \{(x+k+1)^\alpha - (x)^\alpha\}}} \right] \)

Numerator of \( \frac{d(RFR)}{dk} = (1 - e^{-\lambda^k \{(x+k+1)^\alpha - x^\alpha\}}) \)

\[ \left( \alpha \lambda^{\alpha} (x+k+1)^{\alpha-1} e^{-\lambda^k \{(x+k+1)^\alpha - x^\alpha\}} - \alpha \lambda^{\alpha} (x+k)^{\alpha-1} e^{-\lambda^k \{(x+k)^\alpha - x^\alpha\}} \right) < 0 \]

if

\[ (x+k+1)^{\alpha-1} e^{-\lambda (x+k+1)^\alpha} < (x+k)^{\alpha-1} e^{-\lambda (x+k)^\alpha}. \]

This happens if \( l(k) = (x+k)^{\alpha-1} e^{-\lambda (x+k)^\alpha} \) is a decreasing function of \( k. \)

Consider \( \frac{dl(k)}{dk} = (x+k)^{\alpha-2} e^{-\lambda (x+k)^\alpha} \left\{ (\alpha - 1) - \alpha \lambda^{\alpha} (x+k)^\alpha \right\}. \)

For \( \alpha < 1 \) and \( \lambda > 0, \frac{dl(k)}{dk} < 0 \) and

for \( \alpha > 1, \frac{dl(k)}{dk} < 0 \) if \( (\alpha - 1) < \alpha \lambda^{\alpha} (x+k)^\alpha \), that is if \( \lambda > \left( \frac{\alpha - 1}{\alpha (x+k)^\alpha} \right) \frac{1}{\alpha}. \)

Denominator of \( \frac{d(RFR)}{dk} = \left( 1 - e^{-\lambda^k \{(x+k+1)^\alpha - (x)^\alpha\}} \right)^2 > 0. \)
Therefore \( \frac{d(RFR)}{dk} < 0 \)

(a) for \( \alpha < 1 \) and \( \lambda > 0 \);

(b) for \( \alpha > 1 \) and \( \lambda > \left( \frac{\alpha - 1}{\alpha(x+k)^\alpha} \right) \frac{1}{\alpha} \).

This gives the required conclusion.

(iii) For \( \alpha < (>) 1 \), \( \sum_{j=0}^{k} (1 - e^{-\lambda^\alpha[(x+j+1)^\alpha - (x+j)^\alpha]} \),

being the sum of \((k+1)\) decreasing (increasing) functions, is a (an) decreasing

(increasing) function.

For \( \alpha < 1 \), FRA is the product of two positive decreasing functions. So \( K(x) \)

has DFRA.

For \( \alpha > 1 \), it is difficult to conclude as FRA is the product of a decreasing

and an increasing function. By considering few parametric combinations for

\( \alpha > 1 \) and \( \lambda > 0 \), the values of FRA are given evaluated in Table 3. It

is obvious from values in Table 3 that FRA is decreasing (increasing) for

\( \alpha < (>) 1 \) and all \( \lambda > 0 \).

**Proof of Theorem 4**

Let \( f(k) = \frac{(x+k)^\theta}{(x+k+1)^\theta} \).

\[
\frac{df(k)}{dk} = \frac{\theta(x+k)^{\theta-1}(x+k+1)^\theta - \theta(x+k+1)^{\theta-1}(x+k)^\theta}{(x+k+1)^{2\theta}} = \frac{\theta(x+k)^{\theta-1}}{(x+k+1)^{\theta+1}} > 0.
\]

Consider \( g(k) = \left( \frac{(x+k+1)^\theta - (x+k)^\theta}{(x+k+1)^\theta - x^\theta} \right) \left( \frac{x^\theta}{(x+k)^\theta} \right) \)

\[
\frac{dg(k)}{dk} = -\theta \left[ \frac{(x+k+1)^{\theta-1}(x+k)^{\theta-1} - (k+1)(x+k+1)^{\theta-1}x^{\theta-1} + k(x+k)^\theta x^{\theta-1}}{(x+k)^\theta - x^\theta} \right]^2.
\]
\[
\frac{x^\theta}{(x+k)^\theta} + \frac{(x+k+1)^\theta - (x+k)^\theta}{(x+k+1)^\theta - x^\theta} \left( \frac{x^\theta}{(x+k)^{\theta+1}} \right) < 0
\]

Hence \( f(k) \) is increasing and \( g(k) \) is decreasing in \( k \) for \( k \geq 0 \).

(i) Since \( FR = 1 - f(k) \), hence \( FR \) is a decreasing function of \( k \) for \( k \geq 0 \).

(ii) \( RFR = g(k) \) implies that Curtate Pareto is DRFR.

(iii) \( FRA = \frac{1}{k+1} \sum_{j=0}^{k} (1 - f(j)) \).

Since each \( f(j) \) is increasing for \( j = 0, 1, \ldots, k \), hence \( \sum_{j=0}^{k} (1 - f(j)) \), being the sum of \( k \) decreasing functions will be decreasing.

Since \( \frac{1}{k+1} \) is decreasing in \( k \) hence \( FRA \) being the product of two decreasing functions is decreasing in \( k \).

(iv) Since practically \( k \) can take values only upto 100 years, hence we write

\[
MRL = \sum_{j=k+1}^{100} \frac{(x+k)^\theta}{(x+j)^\theta}
\]

\[
= (x+k)^\theta \left( \frac{1}{(x+k+1)^\theta} + \frac{1}{(x+k+2)^\theta} + \ldots + \frac{1}{(x+100)^\theta} \right).
\]

\[
\frac{d}{dk} \left( \frac{(x+k)^\theta}{(x+k+1)^\theta} \right) = \frac{\theta (x+k)^{\theta-1}}{(x+k+1)^{\theta+1}} > 0 \text{ for all } \theta.
\]

This gives that \( \frac{(x+k)^\theta}{(x+k+1)^\theta} \) is an increasing function of \( k \).

It can be shown similarly that \( \frac{(x+k)^{\theta}}{(x+k+2)^{\theta}} \) is an increasing function of \( k \).

Hence for \( \theta \), \( \frac{(x+k)^{\theta}}{(x+j)^{\theta}} \) for \( j = k+1, k+2, \ldots, 100 \) are increasing functions of \( k \). This implies that MRL, being the sum of a finite number of increasing (non-decreasing) functions of \( k \), is increasing for all \( \theta \).

(v) \( K(x) \) will be NWU if \( D_1 = \left( \frac{x}{x+k+j} \right)^\theta - \frac{x^{\theta}}{(x+j)^{\theta}(x+k)^{\theta}} > 0 \);

that is if \( \frac{1}{x^\theta(x+j+k)^{\theta}} > \frac{1}{(x+j)^{\theta}(x+k)^{\theta}} \) which is true for all values of \( \theta \).
and $j, k \geq 0$.

(vi) Similarly consider the difference

$$D_2 = \sum_{j=k}^{\infty} \left( \frac{x}{x+j+k} \right)^{\theta} - \left( \frac{x}{x+k} \right)^{\theta} \sum_{j=k}^{\infty} \left( \frac{x}{x+j} \right)^{\theta} \sum_{j=k}^{\infty} \left( \frac{x}{x+j} \right)^{\theta} \sum_{j=k}^{\infty} \left( \frac{x}{x+j} \right)^{\theta}.$$

From (v), we have

$$x^{\theta} \frac{(x+j+k)^{\theta}}{(x+j)^{\theta}} \frac{(x+j+k)^{\theta}}{(x+k)^{\theta}} > 0$$

for all $\theta > 0$ and $k \geq 0$.

This implies that

$$\sum_{j=k}^{\infty} \frac{x^{\theta}}{(x+j+k)^{\theta}} > \sum_{j=k}^{\infty} \frac{x^{2\theta}}{(x+j)^{\theta} (x+k)^{\theta}}.$$

Hence $D_2 > 0$. This implies that $K(x)$ is NWUE for all $\theta$.

Tables 1 – 9 displaying values of FR, RFR, FRA and MRL for different curtate distributions are presented now.
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### Table 2: Reversed Failure Rate of Curtate Weibull Distribution

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Table 3: Failure Rate Average of Curtate Weibull Distribution

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Table 4: Failure Rate of Curtate Pareto Distribution

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