Moments of Power Function Distribution Based on Ordered Random Variables and Characterization

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ABSTRACT

In this paper simple expressions for single and product moments of generalized order statistics from the power function distribution have been obtained. The results for order statistics and records are deduced from the relations derived. Further, a characterizing result of this distribution on using the conditional moments of the generalized order statistics is discussed.

Keywords: Generalized order statistics, order statistics, records, power function distribution, moments and characterization.

1. Introduction

The concept of generalized order statistics (gos) was introduced by Kamps (1995). Several models of ordered random variables such as order statistics, record values, sequential order statistics, progressive type II censored order statistics and Pfeifer's record values can be discussed as special cases of the gos. Suppose $X(1,n,m,k),\ldots,X(n,n,m,k)$, $(k \geq 1,\ m$ is a real number), are $n\ gos$ from an absolutely continuous distribution function $(df)\ F(x)$ with probability density function $(pdf)\ f(x)$, if their joint pdf is of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[1-F(x_i)]^m\ f(x_i)\right)(1-F(x_n))^{k-1}\ f(x_n) \qquad (1)$$

on the cone $F^{-1}(0) < x_1 \leq \ldots \leq x_n < F^{-1}(1)$, where $\gamma_j = k + (n-j)(m+1) > 0$ for all j, $1 \le j \le n$, k is a positive integer and $m \ge -1$.

If $m = 0$ and $k = 1$, then this model reduces to the ordinary $r-$th order statistic and (1) will be the joint pdf of n order statistics $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ from df $F(x)$. If $k = 1$ and $m = -1$, then (1) will be the joint pdf of the first n record values of the identically and independently distributed (iid) random variables with df $F(x)$ and corresponding pdf $f(x)$.

In view of (1), the marginal pdf of the $r-$th gos, $X(r,n,m,k)$, $1 \le r \le n$, is

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!} [\bar{F}(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)), \tag{2}$$

and the joint pdf of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \le r < s \le n$, is

$$f_{X(r,n,m,k)\,X(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!} [\bar{F}(x)]^m f(x) g_m^{r-1}(F(x))$$

$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\bar{F}(y)]^{\gamma_s - 1} f(y), \quad x < y, \tag{3}$$

where

$$\bar{F}(x) = 1 - F(x), \qquad C_{r-1} = \prod_{i=1}^{r} \gamma_i$$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}(1-x)^{m+1}, & m \ne -1 \\ -\ln(1-x), & m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(0), \quad x \in [0,1).$$

Several authors utilized the concept of gos in their work. References may be made to Kamps and Gather (1997), Keseling (1999), Cramer and Kamps (2000), Ahsanullah (2000, 2004), Habibullah and Ahsanullah (2000), Pawlas and Szynal (2001), Raqab (2001), Kamps and Cramer (2001), Ahmad and Fawzy (2003), Bieniek and Szynal (2003), Al-Hussaini and Ahmad (2003), Saran and Pandey (2003), Cramer *et al.* (2004), Jaheen (2005), Al-Hussaini, *et al.* (2005) Khan *et al.* (2006), Khan *et al.* (2007), Ahmad (2007, 2008), Khan *et al.* (2010) among others.

In this paper we have obtained simple expressions for the exact moments of generalized order statistics from the power function distribution. Results for order statistics and record values are deduced as special cases and a characterization of this distribution is obtained by using conditional moments of generalized order statistics.

A random variable X is said to have the three parameter power function distribution if its probability density function (pdf) is of the following form

$$f(x)=\frac{\alpha}{\sigma}\left(\frac{\mu+\sigma-x}{\sigma}\right)^{\alpha-1}, \mu<x<\mu+\sigma,\ -\infty<\mu<\infty,\ \sigma>0,\ \ \alpha>0.$$

This is a Pearson's Type-I distribution. If $\alpha=1$ then the power function distribution coincides with the uniform distribution on the interval $(\mu,\ \mu+\sigma)$. We will consider in this paper without any loss of generality $\mu=0$ and $\sigma=1$, i.e.

$$f(x)=\alpha(1-x)^{\alpha-1},\quad 0<x<1, \alpha>0 \tag{4}$$

and the corresponding distribution function (df)

$$F(x)=1-(1-x)^{\alpha},\quad 0<x<1,\ \ \alpha>0. \tag{5}$$

For applications of the distribution one may refer to Meniconi and Barry (1996), Zaka and Akhter (2013) and Arslan (2014).

2. Single Moments

Lemma 2.1

For the power function distribution as given in (5) and any non-negative finite integers a and b, when $m\neq -1$

$$J_j(a,b)=\frac{1}{(m+1)^b}\sum_{p=0}^{j}\sum_{u=0}^{b}(-1)^{u+p}\binom{j}{p}\binom{b}{u}\frac{1}{[a+u(m+1)+1+(p/\alpha)]}, \tag{6}$$

for $m=-1$

$$J_j(a,b)=b!\sum_{p=0}^{j}(-1)^{p}\binom{j}{p}\frac{1}{[a+1+(p/\alpha)]^{b+1}}, \tag{7}$$

where

$$J_j(a,b)=\int_0^1 x^j [\bar{F}(x)]^a f(x)\, g_m^b(F(x))\, dx\,. \tag{8}$$

Proof. On expanding binomially the term $g_m^b(F(x))=\left[\frac{1}{m+1}\{1-(\bar{F}(x))^{m+1}\}\right]^b$ in (8), we get

$$J_j(a,b)=A\int_0^1 x^j [\bar{F}(x)]^{a+u(m+1)} f(x)\; dx\,, \quad m\neq -1\,, \tag{9}$$

where

$$A=\frac{1}{(m+1)^b}\sum_{u=0}^{b}(-1)^u\binom{b}{u}.$$

Setting $z=[\bar{F}(x)]^{1/\alpha}$ in (9), we get

$$J_j(a,b)=\alpha\, A\sum_{p=0}^{j}(-1)^p\binom{j}{p}\int_0^1 z^{\alpha[a+u(m+1)+1]+p-1}dz$$

and hence the result.

When $m=-1$ the expression (6) is undefined, therefore we have to consider the limit as m tends to -1. Write

$$J_j(a,b)=A_1\sum_{u=0}^{b}(-1)^u\binom{b}{u}\frac{[a+u(m+1)+1+(p/\alpha)]^{-1}}{(m+1)^b}\,, \tag{10}$$

where

$$A_1=\sum_{p=0}^{j}(-1)^p\binom{j}{p}.$$

Differentiating numerator and denominator of (10) b times with respect to m, we get

$$J_j(a,b)=A_1\sum_{u=0}^{b}(-1)^{u+b}\binom{b}{u}\frac{u^b}{[a+u(m+1)+1+(p/\alpha)]^{b+1}}\,, \quad b>0\,.$$

On applying the L' Hospital rule, we have

$$\lim_{m\to -1} J_j(a,b)=A_1\sum_{u=0}^{b}(-1)^{u+b}\binom{b}{u}\frac{u^b}{[a+1+(p/\alpha)]^{b+1}}\,. \tag{11}$$

But for all integers $n\geq 0$ and for all real numbers x, from Ruiz (1996)

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$$\sum_{i=0}^{n}(-1)^{i}\binom{n}{i}(x-i)^{n}=n!\,. \tag{12}$$

Therefore,

$$\sum_{u=0}^{b}(-1)^{u+b}\binom{b}{u}u^{b}=b!\,. \tag{13}$$

Now on substituting (13) in (11), we have the result given in (7).

Theorem 2.1
For the power function distribution as given in (5) and $1\le r\le n$, $k=1,2,\ldots$, $m\ne -1$,

$$E[X^{j}(r,n,m,k)]=\frac{C_{r-1}}{(r-1)!}J_{j}(\gamma_{r}-1,\ r-1) \tag{14}$$

$$=\frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{p=0}^{j}\sum_{u=0}^{r-1}(-1)^{u+p}\binom{j}{p}\binom{r-1}{u}\frac{1}{[\gamma_{r-u}+(p/\alpha)]}. \tag{15}$$

Proof. From (2) and (8), we have

$$E[X^{j}(r,n,m,k)]=\frac{C_{r-1}}{(r-1)!}J_{j}(\gamma_{r}-1,\ r-1)\,.$$

Making use of Lemma 2.1, we establish the relation given in (15).

Identity 2.1
For $\gamma_r\ge 1$, $k\ge 1$, $1\le r\le n$ and $m\ne -1$

$$\sum_{u=0}^{r-1}(-1)^{u}\binom{r-1}{u}\frac{1}{\gamma_{r-u}}=\frac{(r-1)!(m+1)^{r-1}}{\prod_{t=1}^{r}\gamma_{t}}\,. \tag{16}$$

Proof. (16) can be proved by setting $j=0$ in (15).

Special cases

i) Setting $m=0$, $k=1$ in (15), the explicit formula for the single moments of order statistics of the power function distribution can be obtained as

$$E[X_{r:n}^{j}]=C_{r:n}\sum_{p=0}^{j}\sum_{u=0}^{r-1}(-1)^{u+p}\binom{j}{p}\binom{r-1}{u}\frac{1}{[n-r+1+u+(p/\alpha)]}, \tag{17}$$

where

$$C_{r:n}=\frac{n!}{(r-1)!(n-r)!}.$$

ii) Putting $m=-1$ in (15), we deduce the explicit expression for the single moments of upper k record values for the power function distribution in view of (7) and (14) in the form

$$E[X^{j}(r,n,-1,k)]=E[(Y_r^{(k)})^{j}]=k^{r}\sum_{p=0}^{j}(-1)^{p}\binom{j}{p}\frac{1}{[k+(p/\alpha)]^{r}}$$

and hence for upper records

$$E[(Y_r^{(1)})^{j}]=E[X_{U(r)}^{j}]=\sum_{p=0}^{j}(-1)^{p}\binom{j}{p}\frac{1}{[1+(p/\alpha)]^{r}}. \tag{18}$$

3. Product Moments

Lemma 3.1

For the power function distribution as given in (2) and non-negative integers a, b and c

$$J_{i,j}(a,0,c)=\sum_{p=0}^{j}\sum_{q=0}^{i}(-1)^{p+q}\binom{j}{p}\binom{i}{q}$$
$$\times\frac{1}{[c+1+(p/\alpha)][a+c+2+(p+q)/\alpha]}, \tag{19}$$

where

$$J_{i,j}(a,b,c)=\int_0^1\int_x^1 x^{i}y^{j}[\overline{F}(x)]^{a}f(x)[h_m(F(y))-h_m(F(x))]^{b}$$
$$\times[\overline{F}(y)]^{c}f(y)dydx. \tag{20}$$

Proof. From (20), (5) and (4), we have

$$J_{i,j}(a,0,c)=\alpha\int_0^1 x^{i}(1-x)^{\alpha(a+1)-1}G(x)dx, \tag{21}$$

where

$$G(x)=\alpha\int_x^1 y^{j}(1-y)^{\alpha(c+1)-1}dy. \tag{22}$$

By setting $1-y=z$ in (22), we get

$$G(x)=\sum_{p=0}^{j}(-1)^{p}\binom{j}{p}\frac{(1-x)^{\alpha[c+1+(p/\alpha)]}}{[c+1+(p/\alpha)]}.$$

On substituting the above expression of $G(x)$ in (21), we find that

$$J_{i,j}(a,0,c)=\sum_{p=0}^{j}(-1)^{p}\binom{j}{p}\frac{\alpha}{[c+1+(p/\alpha)]} \times\int_{0}^{1}x^{i}(1-x)^{\alpha[a+c+2+(p/\alpha)]-1}dx. \tag{23}$$

Again by setting $t=1-x$ in (23) and simplifying the resulting expression, we derive the relation given in (19).

Lemma 3.2

For the condition as stated in Lemma 3.1, and $J_{i,j}(a,b,c)$ is as given in (20).

$$J_{i,j}(a,b,c)=\frac{1}{(m+1)^{b}}\sum_{v=0}^{b}(-1)^{v}\binom{b}{v} \times J_{i,j}(a+(b-v)(m+1),0,c+v(m+1)). \tag{24}$$

For $m\neq -1$

$$J_{i,j}(a,b,c)=\frac{1}{(m+1)^{b}}\sum_{p=0}^{j}\sum_{q=0}^{i}\sum_{v=0}^{b}(-1)^{p+q+v}\binom{j}{p}\binom{i}{q}\binom{b}{v} \times\frac{1}{[c+1+v(m+1)+(p/\alpha)][a+c+2+b(m+1)+(p+q)/\alpha]}, \tag{25}$$

and for $m=-1$

$$J_{i,j}(a,b,c)=\sum_{p=0}^{j}\sum_{q=0}^{i}(-1)^{p+q}\binom{j}{p}\binom{i}{q} \times\frac{b!\,\alpha^{b+2}}{[c+1+(p/\alpha)]^{b+1}[a+c+2+(p+q)/\alpha]}. \tag{26}$$

Proof. When $m \neq -1$, we have

$$[h_m(F(y)) - h_m(F(x))]^b = \frac{1}{(m+1)^b}[(\bar{F}(x))^{m+1} - (\bar{F}(y))^{m+1}]^b$$

$$= \frac{1}{(m+1)^b}\sum_{v=0}^{b}(-1)^v \binom{b}{v}[\bar{F}(y)]^{v(m+1)}[\bar{F}(x)]^{(b-v)(m+1)}.$$

Now substituting for $[h_m(F(y)) - h_m(F(x))]^b$ in equation (20), we get

$$J_{i,j}(a,b,c) = \frac{1}{(m+1)^b}\sum_{v=0}^{b}(-1)^v \binom{b}{v} J_{i,j}(a+(b-v)(m+1), 0, c+v(m+1)).$$

Making use of Lemma 3.1, we derive the relation given in (25).

When $m \neq -1$, $J_{i,j}(a,b,c)$ is undefined, as $\sum_{v=0}^{b}(-1)^v \binom{b}{v} = 0$, so after applying L'Hospital rule and (13), (25) can be proved on the lines of (7).

Theorem 3.1

For the power function distribution as given in (5) and $1 \leq r < s \leq n$, $k = 1, 2, \ldots$ with $m \neq -1$,

$$E[X^i(r,n,m,k)\ X^j(s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{r-1}}$$

$$\times \sum_{u=0}^{r-1}(-1)^u \binom{r-1}{u} J_{i,j}(m+u(m+1), s-r-1, \gamma_s - 1), \tag{27}$$

$$= \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{p=0}^{j}\sum_{q=0}^{i}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{p+q+u+v}$$

$$\times \binom{j}{p}\binom{i}{q}\binom{r-1}{u}\binom{s-r-1}{v}\frac{1}{[\gamma_{s-v} + (p/\alpha)][\gamma_{r-u} + (p+q)/\alpha]}. \tag{28}$$

Proof. From (3), we have

$$E[X^i(r,n,m,k)\ X^j(s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^1\int_x^1 x^i y^j [\bar{F}(x)]^m f(x)$$

$$\times g_m^{r-1}(F(x))[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s - 1} f(y)\,dy\,dx. \tag{29}$$

On expanding $g_m^{r-1}(F(x))$ binomially in (29), we get the relation given in (27). Making use of Lemma 3.2, we derive the relation given in (28).

Identity 3.1

For γ_r, $\gamma_s \geq 1$, $k \geq 1$, $1 \leq r < s \leq n$ and $m \neq -1$

$$\sum_{v=0}^{s-r-1} (-1)^v \binom{s-r-1}{v} \frac{1}{\gamma_{s-v}} = \frac{(s-r-1)!(m+1)^{s-r-1}}{\prod_{t=r+1}^{s} \gamma_t}. \tag{30}$$

Proof. At $i = j = 0$ in (28), we have

$$1 = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{u=0}^{r-1} \sum_{v=0}^{s-r-1} (-1)^{u+v} \binom{r-1}{u} \binom{s-r-1}{v} \frac{1}{\gamma_{s-v}\gamma_{r-u}}.$$

Now on using (16), we get the result given in (30).

Special cases

i) Putting $m = 0$, $k = 1$ in (28), the explicit formula for the product moments of order statistics of the power function distribution can be obtained as

$$E[X_{r:n}^i X_{s:n}^j] = C_{r,s:n} \sum_{p=0}^{j} \sum_{q=0}^{i} \sum_{u=0}^{r-1} \sum_{v=0}^{s-r-1} (-1)^{p+q+u+v} \binom{j}{p} \binom{i}{q} \binom{r-1}{u} \binom{s-r-1}{v}$$

$$\times \frac{1}{[n-s+v+1+(p/\alpha)][n-r+u+1+(p+q)/\alpha]},$$

where

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}.$$

ii) Letting m tends to -1 in (28), we deduce the explicit expression for the product moments of upper k record values for the power function distribution in view of (27) and (26) in the form

$$E[(Y_r^{(k)})^i (Y_s^{(k)})^j] = k^s \sum_{p=0}^{j} \sum_{q=0}^{i} (-1)^{p+q} \binom{j}{p} \binom{i}{q}$$

$$\times \frac{1}{[k+(p/\alpha)]^{s-r} [k+(p+q)/\alpha]^r}$$

and hence for upper records

$$E[X^i_{U(r)} X^j_{U(s)}] = \sum_{p=0}^{j}\sum_{q=0}^{i}(-1)^{p+q}\binom{j}{p}\binom{i}{q}\frac{1}{[1+(p/\alpha)]^{s-r}\,[1+(p+q)/\alpha]^r}.$$

Remark 3.1

At $j=0$ in (28), we have

$$E[X^i(r,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{q=0}^{i}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{q+u+v}$$

$$\times\binom{i}{q}\binom{r-1}{u}\binom{s-r-1}{v}\frac{1}{\gamma_{s-v}[\gamma_{r-u}+(q/\alpha)]}. \qquad (31)$$

Making use of (30) in (31) and simplifying the resulting expression, we get

$$E[X^i(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!\,(m+1)^{r-1}}\sum_{q=0}^{i}\sum_{u=0}^{r-1}(-1)^{q+u}$$

$$\times\binom{i}{q}\binom{r-1}{u}\frac{1}{[\gamma_{r-u}+(q/\alpha)]}$$

as obtained in (15).

4. Characterization

Let $X(r,n,m,k)$, $r=1,2,\ldots,n$ be *gos*, then the conditional *pdf* of $X(s,n,m,k)$ given $X(r,n,m,k)=x$, $1\le r<s\le n$, in view of (2) and (3), is

$$f_{X(s,n,m,k)|X(r,n,m,k)}(y\,|\,x) = \frac{C_{s-1}}{(s-r-1)!C_{r-1}}[\bar{F}(x)]^{m-\gamma_r+1}$$

$$\times[(h_m(F(y))-h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}f(y). \qquad (32)$$

Theorem 4.1

Let $X(r,n,m,k)$, $r=1,2,\ldots,n$ be *gos* based on continuous distribution function $F(x)$ with $F(0)=0$ and $0<F(x)<1$ for all $x>0$, then for two consecutive values r and $r+1$, $2\le r+1<s\le n$, the conditional expectation of *gos* $X(s,n,m,k)$ given $X(l,n,m,k)=x$, is given as

$$E[X(s,n,m,k)\,|\,X(l,n,m,k)=x] = 1-(1-x)\prod_{j=1}^{s-l}\left(\frac{\gamma_{l+j}}{\gamma_{l+j}+1/\alpha}\right),$$

$$l=r,\ r+1 \qquad (33)$$

if and only if X has the df

$$\overline{F}(x)=(1-x)^{\alpha}\,,\quad 0<x<1\,,\quad \alpha>0\,.$$

Proof. From (32), we have

$$E[X(s,n,m,k)\,|\,X(r,n,m,k)=x]=\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}$$

$$\times\int_x^1 y\left[1-\left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)^{m+1}\right]^{s-r-1}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)^{\gamma_s-1}\frac{f(y)}{\overline{F}(x)}dy\,. \tag{34}$$

By setting $u=\frac{\overline{F}(y)}{\overline{F}(x)}=\left(\frac{1-y}{1-x}\right)^{\alpha}$ from (5) in (34), we obtain

$$E[X(s,n,m,k)\,|\,X(r,n,m,k)=x]=\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}$$

$$\times\int_0^1[1-(1-x)u^{1/\alpha}]u^{\gamma_s-1}(1-u^{m+1})^{s-r-1}du$$

$$=\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}[I_1-(1-x)I_2], \tag{35}$$

where

$$I_1=\int_0^1 u^{\gamma_s-1}(1-u^{m+1})^{s-r-1}du \tag{36}$$

and

$$I_2=\int_0^1 u^{(1/\alpha)+\gamma_s-1}(1-u^{m+1})^{s-r-1}du\,. \tag{37}$$

Again by setting $t=u^{m+1}$ in (36) and (37), we get

$$I_1=\frac{1}{m+1}\int_0^1 t^{\frac{k}{m+1}+n-s-1}(1-t)^{s-r-1}dt$$

$$=\frac{(m+1)^{s-r-1}\,\Gamma(s-r)}{\prod_{j=1}^{s-r}\gamma_{r+j}}\,.$$

and

$$I_2=\int_0^1 t^{\frac{k+1/\alpha}{\alpha(m+1)}+n-s-1}(1-t)^{s-r-1}dt$$

$$= \frac{(m+1)^{s-r-1}\Gamma(s-r)}{\prod_{j=1}^{s-r}(\gamma_{r+j}+1/\theta)}$$

Substituting these expressions for I_1 and I_2 in (35) and simplifying the resulting expression, we derive the relation in (33).
To prove the sufficient part, we have from (32) and (33)

$$\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}\int_x^1 y[(\overline{F}(x))^{m+1}-(\overline{F}(y))^{m+1}]^{s-r-1}$$
$$\times[\overline{F}(y)]^{\gamma_s-1}f(y)dy=[\overline{F}(x)]^{\gamma_{r+1}}H_r(x)\,, \qquad (38)$$

where

$$H_r(x)=1-(1-x)\prod_{j=1}^{s-r}\left(\frac{\gamma_{r+j}}{\gamma_{r+j}+1/\alpha}\right).$$

Differentiating (38) both sides with respect to x, we get

$$-\frac{C_{s-1}[\overline{F}(x)]^m f(x)}{(s-r-2)!C_{r-1}(m+1)^{s-r-2}}\int_0^x y[(\overline{F}(x))^{m+1}-(\overline{F}(y))^{m+1}]^{s-r-2}$$

$$\times[\overline{F}(y)]^{\gamma_s-1}f(y)dy=H_r'(x)[\overline{F}(x)]^{\gamma_{r+1}}-\gamma_{r+1}H_r(x)[\overline{F}(x)]^{\gamma_{r+1}-1}f(x)$$

or

$$-\gamma_{r+1}H_{r+1}(x)[\overline{F}(x)]^{\gamma_{r+2}+m}f(x)$$
$$=H_r'(x)[\overline{F}(x)]^{\gamma_{r+1}}-\gamma_{r+1}H_r(x)[\overline{F}(x)]^{\gamma_{r+1}-1}f(x).$$

Therefore,

$$\frac{f(x)}{\overline{F}(x)}=-\frac{H_r'(x)}{\gamma_{r+1}[H_{r+1}(x)-H_r(x)]}=\frac{\alpha}{(1-x)}$$

which proves that

$$F(x)=1-(1-x)^{\alpha}\,,\ 0<x<1\,,\ \ \alpha>0\,.$$

Remark 4.1

For $k=1$, $m=0$ and $k=1$, $m=-1$, we obtain the characterization results of the power function distribution based on order statistics and record values, respectively.

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References

1. Ahmad, A.A. (2007). Relations for single and product moments of generalized order statistics from doubly truncated Burr type XII distribution. *Journal of the Egyptian Mathematical Society*, **15**, 117-128. DOI:
2. Ahmad, A.A. (2008). Single and product moments of generalized order statistics from linear exponential distribution. *Communications in Statistics-Theory and Methods*, **37**, 1162-1172. DOI:10.1080/03610920701713344
3. Ahmad, A.A. and Fawzy, A.M. (2003). Recurrence relations for single moments of generalized order statistics from doubly truncated distributions. *Journal of Statistical Planning and Inference*, **117**, 241–249. DOI:10.1016/S0378-3758(02)00385-3
4. Ahsanullah,M. **(**2000). Generalized order statistics from exponential distribution. *Journal of Statistical Planning and Inference*, **85**, 85-91. DOI:10.1016/S0378-3758(99)00068-3
5. Ahsanullah, M. (2004). A characterization of the uniform distribution by dual generalized order statistics. *Communications in Statistics-Theory and Methods*, **33**, 2921-2928. DOI:10.1081/STA-200038854
6. Al-Hussaini, E.K. and Ahmad, A.A. (2003). On Bayesian predictive distributions of generalized order statistics. *Metrika*, **57**, 165-176. DOI:10.1007/s001840200207
7. Al-Hussaini, E.K., Ahmad, A.A. and Al-Kashif, M.A. (2005). Recurrence relations for moment and conditional moment generating functions of generalized order statistics. *Metrika*, **61**, 199-220. DOI:10.1007/s001840400332

8. Arslan, G. (2014). A new characterization of the power distribution. *Journal of Computational and Applied Mathematics*,**260**, 99-102. DOI:10.1016/j.cam.2013.09.068

9. Bieniek, M. and Szynal, D. (2003). Characterizations of distributions via linearity of regression of generalized order statistics. *Metrika*, **58**, 259-271. DOI:10.1007/s001840300263

10. Cramer, E. and Kamps, U. (2000). Relations for expectations of functions of generalized order statistics. *Journal of Statistical Planning and Inference*, **89**, 79-89. DOI:10.1016/S0378-3758(00)00074-4

11. Cramer, E., Kamps, U. and Keseling, C. (2004). Characterization via linear regression of ordered random variables: a unifying approach. *Communications in Statistics-Theory and Methods*, **33**, 2885-2911. DOI:10.1081/STA-200038832

12. Habibullah, M. and Ahsanullah, M. (2000). Estimation of parameters of a Pareto distribution by generalized order statistics. *Communications in Statistics-Theory and Methods*, **29**, 1597-1609. DOI:10.1080/03610920008832567

13. Jaheen, Z.F. (2005). Estimation based on generalized order statistics from the Burr model. *Communications in Statistics-Theory and Methods*, **34**, 785-794. DOI:10.1081/STA-200054408

14. Kamps, U. (1995). *A Concept of Generalized Order Statistics*. B.G. Teubner Stuttgart, Germany. DOI:10.1007/978-3-663-09196-7

15. Kamps, U. and Cramer, E. (2001). On distributions of generalized order statistics. *Statistics*, **35**, 269-280. DOI:10.1080/02331880108802736

16. Kamps, U. and Gather, U. (1997). Characteristic property of generalized order statistics for exponential distribution. *ApplicationesMathematicae*, **24**, 383-391.

17. Keseling, C. (1999). Conditional distributions of generalized order statistics and some characterizations. *Metrika*, **49**, 27-40. DOI:10.1007/s001840050023

18. Khan, R.U., Anwar, Z. and Athar, H. (2007). Recurrence relations for single and product moments of generalized order statistics from doubly truncated Weibull distribution. *The Aligarh Journal of Statistics*, **27**, 69-79.

19. Khan, A.H., Khan, R.U. and Yaqub, M. (2006). Characterization of continuous distributions through conditional expectation of functions of generalized order statistics. *Journal of Applied Probability and Statistics,***1**, 115-131. DOI:10.12785/jsap/010204

20. Khan, R.U., Zia, Benazir and Athar, H. (2010). On moment generating function of generalized order statistics from Gompertz distribution and its characterization. *Journal of Statistical Theory and Application,***9**, 363-373. DOI:

21. Meniconi, M. and Barry, D.M. (1996). The power function distribution: A useful and simple distribution to assess electrical component reliability. *Microelectronics Reliability*, **36**, 1207-1212. DOI:10.1016/0026-2714(95)00053-4

22. Pawlas, P. and Szynal, D. (2001). Recurrence relations for single and product moments of generalized order statistics from Pareto, generalized Pareto, and Burr distributions. *Communications in Statistics-Theory and Methods*, **30**, 739-746. DOI:10.1081/STA-100002148

23. Raqab, M.Z. (2001). Optimal prediction-intervals for the exponential distribution based on generalized order statistics. *IEEE Transactions on Reliability*, **50**, 112-115. DOI:10.1109/24.935025

24. Ruiz, S.M. (1996). An algebraic identity leading to Wilson's theorem. *Gazette des Math´ematiciens*, **80**, 579-582. DOI:10.2307/3618534

25. Saran, J. and Pandey, A. (2003). Recurrence relations for marginal and joint moment generating functions of generalized order statistics from power function distribution. *Metron*, **LXI**,27-33.

26. Zaka, Azam and Akhter, A.S. (2013). Methods for estimating the parameters of the power function distribution. *Pakistan Journal of Statistics and Operation Research*, **9**, 213-224.