

Improved Estimation of Current Population Mean Over Two Occasions

Housila P. Singh and Surya K. Pal*

School of Studies in Statistics,
Vikram University, Ujjain- 456010, (M.P.) India.

*Corresponding Author: suryakantpal6676@gmail.com.

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ABSTRACT

This paper addresses the role of two auxiliary variables on both the occasions to improve the precision of estimates at current (second) occasion in two occasion successive sampling. Utilizing the readily available information on two auxiliary variables on both the occasions and information on the study variable from the previous occasion, an efficient estimation procedure of population mean on current (second) occasion has been envisaged. It is to be mentioned that out of the two auxiliary variables one auxiliary variable is positively correlated with the study variable while other is negatively correlated. Behavior of the proposed estimator has been studied and compared with the sample mean estimator, when there is no matching from the previous occasion and traditional successive sampling estimator, which is a linear combination of the means of the matched and unmatched portions of the sample at the current (second) occasion. Optimal replacement policy is also discussed. In addition, we support the theoretical results with the aid of numerical examples.

Keywords: Successive sampling, Auxiliary variables, Bias, Mean squared error.

1. Introduction

It is well recognized fact that the use of auxiliary information in estimating the parameters such as population mean or total improve the precision of estimates. If the survey is repetitive in nature, past values may also be used as auxiliary information to improve the precision of the current estimates. Jessen (1942) was the first who introduced the procedure of utilizing the information obtained on first occasion in improving the estimates of the current occasion. Estimation on more than two occasions is due to Patterson (1950). The theory of Patterson (1950) has been extended by Eckler (1955), Rao and Graham (1964), Cochran (1977), Sen

(1971, 1973), Sukhatme et al. (1984) and Singh et al. (1992), Singh et al. (2014) among others. In many situations of practical importance, information on an auxiliary variable may be readily available on the first as well as on the second occasion, for instance, see Singh and Vishwakarma (2007a, b, 2009), Singh and Kumar (2010), Kumar (2012) and Singh et al. (2013). The aim of this paper is to develop a procedure of utilizing the information on two auxiliary variables readily available on both the occasions. We have assumed a situation, where one auxiliary variable is positively correlated and the other auxiliary variable is negatively correlated with the study variable on the first and second occasions. In this situation, we have made an effort to propose an estimator of the population mean on current occasion using information on two auxiliary variates on both occasions in two occasion successive sampling besides the information on the study variable on the previous occasion. Theoretical properties of the suggested estimator have been discussed and numerical illustrations are given. Numerical results show the dominance of the suggested estimator over the sample mean estimator and the natural successive sampling estimator when no auxiliary information is used. Suitable recommendations have been made on the basis of numerical findings.

2. Development of the Estimator

Let $U = (U_1, U_2, \dots, U_N)$ be the finite population of N units, which has been sampled over two occasions. The variable under study is designated as $x(y)$ on the first (second) occasion respectively. It is assumed that information on two auxiliary variables z_1 and z_2 (where population means \bar{Z}_1 and \bar{Z}_2 are known) readily available on the first as well as on the second occasion. It is assumed that the information on two auxiliary variable z_1 is positively correlated with x and y while the auxiliary variable z_2 is negatively correlated with x and y . For instance, in estimating the mean yield (y) of rice per plant on current (second) occasion, the two auxiliary variables may be number of tillers (z_1) and percentage of sterility (z_2) such that y is correlated positively with number of tillers (z_1) and negatively with percentage sterility (z_2); see Sahoo and Swain (1980). For more examples the reader is referred to Singh (1967). A simple random sample of n units is drawn without replacement on the first occasion. A random sub-sample of $m = n\lambda$ units is retained (matched) for its use on the second occasion, while a fresh sample of size of $u = (n-m) = n\mu$ units is drawn without replacement on the second occasion from the population of $(N-n)$ units. So that the sample size on the second occasion is n as well. The fractions of matched and fresh sample respectively are denoted by λ and μ such that $\lambda + \mu = 1$.

In what follows, we shall use the following notations:

$\bar{X}, \bar{Y}, \bar{Z}_1, \bar{Z}_2$: The population means of the variables x, y, z_1 and z_2 respectively,

$\bar{x}_n, \bar{x}_m, \bar{y}_u, \bar{y}_m, \bar{z}_{1n}, \bar{z}_{2n}, \bar{z}_{1u}, \bar{z}_{2u}$: The sample means of the respective variables of the sample size shown in suffices,

$\rho_{yx}, \rho_{yz_1}, \rho_{yz_2}, \rho_{xz_1}, \rho_{xz_2}$: The population correlation coefficients between the variables shown in suffices,

$S_x^2 = (N-1)^{-1} \sum_{i=1}^N (x_i - \bar{X})^2$: The population mean square of the variable x ,

$S_y^2, S_{z_1}^2, S_{z_2}^2$: The population mean squares of the variables y, z_1 and z_2 respectively.

$C_y, C_x, C_{z_1}, C_{z_2}$: The population coefficients of variation of the variables y, x, z_1 and z_2 respectively.

Utilizing information on two auxiliary variables, two different estimators of the population mean \bar{Y} on the current (second) occasion may be considered. One estimator is based on the fresh sample of size $u (= n\mu)$ drawn on the second occasion defined by

$$d_u = \bar{y}_u \exp\left(\frac{\bar{Z}_1 - \bar{z}_{1u}}{\bar{Z}_1 + \bar{z}_{1u}}\right) \exp\left(\frac{\bar{z}_{2u} - \bar{Z}_2}{\bar{z}_{2u} + \bar{Z}_2}\right) \quad (1)$$

The motivation in defining the estimator d_u is taken from Bahl and Tuteja (1991). The second estimator is based on the sample of size $m (= n\lambda)$ common to both the occasions given by

$$d_m = \bar{y}_m \left(\frac{\bar{x}_n}{\bar{x}_m}\right) \exp\left(\frac{\bar{Z}_1 - \bar{z}_{1n}}{\bar{Z}_1 + \bar{z}_{1n}}\right) \exp\left(\frac{\bar{z}_{2n} - \bar{Z}_2}{\bar{z}_{2n} + \bar{Z}_2}\right) \quad (2)$$

The estimator d_u may be used to estimate the population mean on each occasion, while the estimator d_m is suitable to estimate the change over occasions. To device suitable estimation procedures for both the problems simultaneously, a convex linear combination of d_u and d_m has been taken as a final estimator of the population mean \bar{Y} and is given by

$$d = \phi d_u + (1 - \phi) d_m, \quad (3)$$

where $\phi (0 \leq \phi \leq 1)$ is a scalar to be determined such that the mean square error (MSE) of d is minimum.

3. Properties of the Proposed Estimator

To obtain the bias (B (.)) and mean square error (MSE (.)) of the estimator d, we write

$$\bar{y}_u = \bar{Y}(1 + e_{yu}) \Rightarrow e_{yu} = (\bar{y}_u - \bar{Y}) / \bar{Y},$$

$$\bar{v}_m = \bar{V}(1 + e_{vm}) \Rightarrow e_{vm} = (\bar{v}_m - \bar{V}) / \bar{V}, v = x, y;$$

$$\bar{x}_n = \bar{X}(1 + e_{xn}) \Rightarrow e_{xn} = (\bar{x}_n - \bar{X}) / \bar{X},$$

$$\bar{z}_{iu} = \bar{Z}_i(1 + e_{iu}) \Rightarrow e_{iu} = (\bar{z}_{iu} - \bar{Z}_i) / \bar{Z}_i, i = 1, 2;$$

$$\bar{z}_{in} = \bar{Z}_i(1 + e_{in}) \Rightarrow e_{in} = (\bar{z}_{in} - \bar{Z}_i) / \bar{Z}_i, i = 1, 2;$$

such that

$$E(e_{yu}) = E(e_{vm}) = E(e_{xn}) = E(e_{iu}) = E(e_{in}) = 0, v = x, y; i = 1, 2.$$

- **Relative Variances**

$$E(e_{yu}^2) = ((1/u) - (1/N))C_y^2,$$

$$E(e_{vm}^2) = ((1/m) - (1/N))C_v^2, v = x, y;$$

$$E(e_{iu}^2) = ((1/u) - (1/N))C_{z_i}^2,$$

$$E(e_{in}^2) = ((1/n) - (1/N))C_{z_i}^2, i = 1, 2;$$

$$E(e_{xn}^2) = ((1/n) - (1/N))C_x^2.$$

- **Relative Covariances**

$$E(e_{yu}e_{xm}) = -(1/N)C_y^2,$$

$$E(e_{yu}e_{xm}) = E(e_{yu}e_{xn}) = -(1/N)\rho_{yx}C_yC_x,$$

$$E(e_{yu}e_{iu}) = ((1/u) - (1/N))\rho_{yz_i}C_yC_{z_i},$$

$$E(e_{ym}e_{in}) = ((1/n) - (1/N))\rho_{yz_i}C_yC_{z_i},$$

$$E(e_{yu}e_{in}) = E(e_{ym}e_{in}) = -(1/N)\rho_{yz_i}C_yC_{z_i},$$

$$E(e_{xm}e_{iu}) = E(e_{xn}e_{iu}) = -(1/N)\rho_{xz_i}C_xC_{z_i},$$

$$E(e_{xm}e_{in}) = E(e_{xn}e_{in}) = ((1/n) - (1/N))\rho_{xz_i}C_xC_{z_i},$$

$$E(e_{iu}e_{in}) = -(1/N)C_{z_i}^2.$$

$$E(e_{ym}e_{xm}) = ((1/m) - (1/N))\rho_{yx}C_yC_x,$$

} , i = 1, 2;

$$\begin{aligned}
 E(e_{ym}e_{xn}) &= ((1/n) - (1/N))\rho_{yx} C_y C_x, \\
 E(e_{1u}e_{2u}) &= ((1/u) - (1/N))\rho_{z_1z_2} C_{z_1} C_{z_2}, \\
 E(e_{1n}e_{2n}) &= ((1/n) - (1/N))\rho_{z_1z_2} C_{z_1} C_{z_2}, \\
 E(e_{1u}e_{2n}) &= E(e_{2u}e_{1n}) = -(1/N)\rho_{z_1z_2} C_{z_1} C_{z_2}.
 \end{aligned}$$

3.1 The Bias and MSE of the Estimator d_u

Expressing d_u in terms of e 's we have

$$\begin{aligned}
 d_u &= \bar{Y}(1 + e_{yu}) \exp\left(\frac{-e_{1u}}{2 + e_{1u}}\right) \exp\left(\frac{e_{2u}}{2 + e_{2u}}\right) \\
 &= \bar{Y}(1 + e_{yu}) \exp\left\{-\frac{e_{1u}}{2} \left(1 + \frac{e_{1u}}{2}\right)^{-1}\right\} \exp\left\{\frac{e_{2u}}{2} \left(1 + \frac{e_{2u}}{2}\right)^{-1}\right\}. \tag{4}
 \end{aligned}$$

Expanding the right hand side of (4), multiplying out and neglecting terms of e 's having power greater than two, we have

$$\begin{aligned}
 d_u &\cong \bar{Y}[1 + e_{yu} + (1/2)(e_{2u} - e_{1u}) + (1/2)(e_{yu}e_{2u} - e_{yu}e_{1u}) \\
 &\quad + (1/8)(3e_{1u}^2 - e_{2u}^2 - 2e_{1u}e_{2u})] \\
 \text{or} \\
 (d_u - \bar{Y}) &\cong \bar{Y}[e_{yu} + (1/2)(e_{2u} - e_{1u}) + (1/2)(e_{yu}e_{2u} - e_{yu}e_{1u}) \\
 &\quad + (1/8)(3e_{1u}^2 - e_{2u}^2 - 2e_{1u}e_{2u})]. \tag{5}
 \end{aligned}$$

Taking expectation of both sides of (5) we get the bias of d_u to the first degree of approximation as

$$\begin{aligned}
 B(d_u) &= \bar{Y}((1/u) - (1/N))[(1/2)(\rho_{yz_2} C_y C_{z_2} - \rho_{yz_1} C_y C_{z_1}) \\
 &\quad + (1/8)(3C_{z_1}^2 - C_{z_2}^2 - 2\rho_{z_1z_2} C_{z_1} C_{z_2})] \tag{6}
 \end{aligned}$$

Remark - 3.1 Since x and y denote the same study variable over two occasions, z_1 and z_2 are the stable auxiliary variables correlated the study variable $x(y)$, therefore, looking on the stability nature of the coefficients of variation [Murthy (1967, p.325)] and following Cochran (1977) and Feng and Zou (1977), the coefficients of variation of the variables x, y, z_1 and z_2 are considered to be approximately equal (i.e. $C_x \cong C_{z_1} \cong C_{z_2} \cong C_y$). Thus under the Remark 3.1 the expression (6) reduces to:

$$\begin{aligned}
 B(d_u) &= \bar{Y}[(1/2u)\{\theta + (1/2)(1 - \rho_{z_1 z_2})\} - (1/2N)\{\theta + (1/2)(1 - \rho_{z_1 z_2})\}]C_y^2 \\
 &= (\bar{Y}/2)[(1/u)\{\theta + (1/2)(1 - \rho_{z_1 z_2})\} - (1/N)\{\theta + (1/2)(1 - \rho_{z_1 z_2})\}]C_y^2, \quad (7)
 \end{aligned}$$

where $\theta = (\rho_{yz_2} - \rho_{yz_1})$.

Squaring both sides of (7) and neglecting terms of e's having power greater than two we have

$$(d_u - \bar{Y})^2 \cong \bar{Y}^2[e_{yu}^2 + (1/4)(e_{2u}^2 + e_{1u}^2 - 2e_{1u}e_{2u}) + (e_{yu}e_{2u} - e_{yu}e_{1u})], \quad (8)$$

Taking expectation of both sides of (8) we get the MSE of d_u to the first degree of approximation as

$$\begin{aligned}
 MSE(d_u) &= ((1/u) - (1/N))\bar{Y}^2[C_y^2 + (1/4)(C_{z_1}^2 + C_{z_2}^2 - 2\rho_{z_1 z_2}C_{z_1}C_{z_2}) \\
 &\quad + (\rho_{yz_2}C_yC_{z_2} - \rho_{yz_1}C_yC_{z_1})] \quad (9)
 \end{aligned}$$

Under the Remark 3.1 the $MSE(d_u)$ in (9) reduces to:

$$\begin{aligned}
 MSE(d_u) &= ((1/u) - (1/N))S_y^2[(3/2) + \theta - (1/2)\rho_{z_1 z_2}] \\
 &= ((1/u) - (1/N))S_y^2\delta_3 \quad (10)
 \end{aligned}$$

where $\delta_3 = [(3/2) + \theta - (1/2)\rho_{z_1 z_2}]$.

Thus we have established the following theorem.

Theorem 3.1 To the first degree of approximation, under Remark 3.1, the bias and MSE of d_u are respectively given by

$$B(d_u) = (\bar{Y}C_y^2/2)((1/u) - (1/N))[\theta + (1/2)(1 - \rho_{z_1 z_2})] \quad (11)$$

and

$$MSE(d_u) = ((1/u) - (1/N))S_y^2\delta_3. \quad (12)$$

3.2 The Bias and MSE of the Estimator d_m

Expressing d_m in terms of e's we have

$$\begin{aligned}
 d_m &= \bar{Y}(1 + e_{ym})(1 + e_{xm})^{-1}(1 + e_{xn})\exp\left(\frac{-e_{1n}}{2 + e_{1n}}\right)\exp\left(\frac{e_{2n}}{2 + e_{2n}}\right) \\
 &= \bar{Y}(1 + e_{ym})(1 + e_{xm})^{-1}(1 + e_{xn})\exp\left\{-\frac{e_{1n}}{2}\left(1 + \frac{e_{1n}}{2}\right)^{-1}\right\}\exp\left\{\frac{e_{2n}}{2}\left(1 + \frac{e_{2n}}{2}\right)^{-1}\right\}, \quad (13)
 \end{aligned}$$

Expanding the right hand side of (13) multiplying out and neglecting terms of e 's having power greater than two we have

$$d_m \cong \bar{Y}[1 + e_{ym} - e_{xm} + e_{xn} + (1/2)(e_{2n} - e_{1n}) - e_{ym}e_{xm} + e_{ym}e_{xn} - e_{xm}e_{xn} + e_{xm}^2 + (1/2)(e_{ym}e_{2n} + e_{xn}e_{2n} - e_{xm}e_{2n}) - (1/8)e_{2n}^2 - (1/2)(e_{ym}e_{1n} + e_{xn}e_{1n} - e_{xm}e_{1n} + e_{1n}e_{2n}) + (3/8)e_{1n}^2]$$

or

$$(d_m - \bar{Y}) \cong \bar{Y}[e_{ym} - e_{xm} + e_{xn} + (1/2)(e_{2n} - e_{1n}) - e_{ym}e_{xm} + e_{ym}e_{xn} - e_{xm}e_{xn} + e_{xm}^2 + (1/2)(e_{ym}e_{2n} + e_{xn}e_{2n} - e_{xm}e_{2n}) - (1/8)e_{2n}^2 - (1/2)(e_{ym}e_{1n} + e_{xn}e_{1n} - e_{xm}e_{1n} + e_{1n}e_{2n}) + (3/8)e_{1n}^2]. \tag{14}$$

Taking expectation of both sides of (14) we get the bias of d_m to the first degree of approximation as

$$B(d_m) = \bar{Y}[(1/m) - (1/n)](C_x^2 - \rho_{yx}C_yC_x) + [(1/n) - (1/N)] \times \{(1/2)(\rho_{yz}C_yC_z - \rho_{yz}C_yC_z) + (1/8)(3C_{z_1}^2 - C_{z_2}^2 - 4\rho_{z_1z_2}C_{z_1}C_{z_2})\}, \tag{15}$$

Under the Remark 3.1 the bias of d_m in (15) reduces to:

$$B(d_m) = \bar{Y}C_y^2 [(1/m) - (1/n)](1 - \rho_{yx}) + [(1/n) - (1/N)]\{(1/2)\theta + (1/4)(1 - 2\rho_{z_1z_2})\} = \bar{Y}C_y^2 [(1/m)(1 - \rho_{yx}) + (1/n)\{-3/4 + \rho_{yx} + (1/2)\theta - (1/2)\rho_{z_1z_2}\} - (1/N)\{(1/2)\theta + (1/4)(1 - 2\rho_{z_1z_2})\}]. \tag{16}$$

Squaring both sides of (14) and neglecting terms of e 's having power greater than two, we have

$$(d_m - \bar{Y})^2 \cong \bar{Y}^2[e_{ym}^2 + (e_{xm} - e_{xn})^2 + (1/4)(e_{2n} - e_{1n})^2 - 2(e_{ym}e_{xm} - e_{ym}e_{xn}) + (e_{ym}e_{2n} - e_{ym}e_{1n}) - (e_{xm}e_{2n} - e_{xm}e_{1n}) + (e_{2n}e_{xn} - e_{1n}e_{xn})]. \tag{17}$$

Taking expectation of both sides of (17) we get the MSE of d_m to the first degree of approximation as

$$MSE(d_m) = \bar{Y}^2[(1/m) - (1/N)]C_y^2 + [(1/m) - (1/n)](C_x^2 - 2\rho_{yx}C_yC_x) + [(1/n) - (1/N)]\{(1/4)(C_{z_1}^2 + C_{z_2}^2 - 2\rho_{z_1z_2}C_{z_1}C_{z_2}) + (\rho_{yz}C_yC_z - \rho_{yz}C_yC_z)\}. \tag{18}$$

Under the Remark 3.1 the $MSE(d_m)$ in (18) reduces to:

$$\begin{aligned}
 MSE(d_m) &= S_y^2 \left[\left(\frac{1}{m} - \frac{1}{N} \right) + \left(\frac{1}{m} - \frac{1}{n} \right) (1 - 2\rho_{yx}) \right. \\
 &\quad \left. + \left(\frac{1}{n} - \frac{1}{N} \right) \left\{ \left(\frac{1}{2} \right) (1 - \rho_{z_1 z_2}) + \theta \right\} \right] \\
 &= S_y^2 \left[\left(\frac{1}{m} \right) (2 - 2\rho_{yx}) + \left(\frac{1}{n} \right) \left\{ -\left(\frac{1}{2} \right) + \theta - \left(\frac{1}{2} \right) \rho_{z_1 z_2} + 2\rho_{yx} \right\} \right. \\
 &\quad \left. - \left(\frac{1}{N} \right) \left\{ \left(\frac{3}{2} \right) + \theta - \left(\frac{1}{2} \right) \rho_{z_1 z_2} \right\} \right] \\
 &= S_y^2 \left[\left(\frac{1}{m} \right) \alpha_1 + \left(\frac{1}{n} \right) \alpha_2 - \left(\frac{1}{N} \right) \alpha_3 \right], \tag{19}
 \end{aligned}$$

where $\alpha_1 = \{2 - 2\rho_{yx}\}$, $\alpha_2 = \{-\frac{1}{2} + \theta - \frac{1}{2}\rho_{z_1 z_2} + 2\rho_{yx}\}$ and $\alpha_3 = \{\frac{3}{2} + \theta - \frac{1}{2}\rho_{z_1 z_2}\}$

Now we state the following theorem.

Theorem 3.2 Under Remark 3.1, the bias and MSE of d_m to the first degree of approximation, are respectively given by

$$\begin{aligned}
 B(d_m) &= \bar{Y} C_y^2 \left[\left(\frac{1}{m} \right) (1 - \rho_{yx}) + \left(\frac{1}{n} \right) \left\{ -\left(\frac{3}{4} \right) + \rho_{yx} + \left(\frac{1}{2} \right) \theta - \left(\frac{1}{2} \right) \rho_{z_1 z_2} \right\} \right. \\
 &\quad \left. - \left(\frac{1}{N} \right) \left\{ \left(\frac{1}{4} \right) + \left(\frac{1}{2} \right) \theta - \left(\frac{1}{2} \right) \rho_{z_1 z_2} \right\} \right] \tag{20}
 \end{aligned}$$

and

$$MSE(d_m) = S_y^2 \left[\left(\frac{1}{m} \right) \alpha_1 + \left(\frac{1}{n} \right) \alpha_2 - \left(\frac{1}{N} \right) \alpha_3 \right]. \tag{21}$$

Now, we derive the expression for covariance between the two estimators d_u and d_m as

$$\begin{aligned}
 Cov(d_u, d_m) &= E[(d_u - \bar{Y})(d_m - \bar{Y})] \tag{22} \\
 &= \bar{Y}^2 E \left[\{e_{yu} - \frac{1}{2}(e_{1u} - e_{2u})\} \{e_{ym} - (e_{xm} - e_{xn}) - \frac{1}{2}(e_{1n} - e_{2n})\} \right] \\
 &= \bar{Y}^2 E \left[e_{yu} e_{ym} - (e_{yu} e_{xm} - e_{yu} e_{xn}) - \frac{1}{2}(e_{yu} e_{1n} - e_{yu} e_{2n}) \right. \\
 &\quad \left. - \frac{1}{2}(e_{ym} e_{1u} - e_{ym} e_{2u}) + \frac{1}{2}(e_{1u} e_{xm} - e_{1u} e_{xn} - e_{2n} e_{xm} + e_{2n} e_{xn}) \right. \\
 &\quad \left. + \frac{1}{4}(e_{1u} e_{1n} - e_{1u} e_{2n} - e_{2u} e_{1n} + e_{2u} e_{2n}) \right] \\
 &= -(\bar{Y}^2 / N) (C_y^2 + (\rho_{yz_2} C_y C_{z_2} - \rho_{yz_1} C_y C_{z_1}) \\
 &\quad + \frac{1}{4}(C_{z_1}^2 + C_{z_2}^2 - 2\rho_{z_1 z_2} C_{z_1} C_{z_2}) \}. \tag{23}
 \end{aligned}$$

Under the Remark 3.1, the expression (23) reduces to:

$$\begin{aligned} \text{Cov}(d_u, d_m) &= -(S_y^2 / N) \{ (3/2) + \theta - (1/2) \rho_{z_1 z_2} \} \\ &= -(S_y^2 / N) \{ \alpha_1 + \alpha_2 \}. \end{aligned} \quad (24)$$

3.3. The Bias and MSE of the Combined Estimator ‘d’

Under the Remark 3.1, we state the following theorems.

Theorem 3.3 Bias of the estimator ‘d’ to the first degree of approximation, is given by

$$B(d) = \phi B(d_u) + (1 - \phi) B(d_m), \quad (25)$$

where $B(d_u)$ and $B(d_m)$ are respectively given by (7) and (16).

For proof see Appendix-I.

Theorem 3.4 The mean squared error of ‘d’ to the first degree of approximation is given by

$$MSE(d) = \phi^2 MSE(d_u) + (1 - \phi)^2 MSE(d_m) + 2\phi(1 - \phi) \text{Cov}(d_u, d_m), \quad (26)$$

where $MSE(d_u)$, $MSE(d_m)$ and $\text{Cov}(d_u, d_m)$ are respectively given by (10), (18) and (24).

For proof see Appendix-I.

3.4. Minimum Mean Square Error of the Estimator ‘d’

Since the mean square error of the estimator ‘d’ in (26) is a function of unknown scalar ϕ , therefore, it is minimized with respect to ϕ and subsequently the optimum value of ϕ , say ϕ_{opt} is obtained as

$$\phi_{opt} = \frac{[\text{MSE}(d_m) - \text{Cov}(d_u, d_m)]}{[\text{MSE}(d_u) + \text{MSE}(d_m) - 2\text{Cov}(d_u, d_m)]}. \quad (27)$$

Substitution of the optimum value of ϕ in (26) yields the optimum mean squared error of the estimator ‘d’ as

$$\text{MSE}(d)_{opt} = \frac{[\text{MSE}(d_u)\text{MSE}(d_m) - \{\text{Cov}(d_u, d_m)\}^2]}{[\text{MSE}(d_u) + \text{MSE}(d_m) - 2\text{Cov}(d_u, d_m)]}. \quad (28)$$

Further, putting the values of $MSE(d_u)$, $MSE(d_m)$ and $Cov(d_u, d_m)$ from the equations (10), (18) and (24) in (27) and (28) we get the simplified value of Φ_{opt} and $MSE(d)_{opt}$ respectively as

$$\Phi_{opt} = \frac{[\mu(\alpha_1 + \alpha_2) - \mu^2\alpha_2]}{(\alpha_3 + \mu\alpha_7 + \mu^2\alpha_8)} \tag{29}$$

and

$$MSE(d)_{opt} = \frac{(\alpha_4 + \mu\alpha_5 + \mu^2\alpha_6) S_y^2}{(\alpha_3 + \mu\alpha_7 + \mu^2\alpha_8) n} \tag{30}$$

where $\alpha_3 = (\alpha_1 + \alpha_2) = [(3/2) + \theta - (1/2)\rho_{z_1z_2}]$, $\alpha_4 = (\alpha_1\alpha_3 + \alpha_2\alpha_3 - f\alpha_3^2)$, $\alpha_5 = (-\alpha_1\alpha_3f - \alpha_2\alpha_3 - \alpha_2\alpha_3f + f\alpha_3^2)$, $\alpha_6 = \alpha_2\alpha_3f$, $\alpha_7 = (\alpha_1 + \alpha_2 - \alpha_3)$, $\alpha_8 = -\alpha_2$ and $f = n/N$.

3.5. Optimum Replacement Policy

To obtain the optimum value of μ (fraction of sample to be drawn a fresh on second occasion) so that the population mean \bar{Y} may be estimated with maximum precision, we minimize the equation (30) with respect to μ , which yields a quadratic equation in μ :

$$A_1\mu^2 + 2A_2\mu + A_3 = 0, \tag{31}$$

where $A_1 = (\alpha_6\alpha_7 + \alpha_2\alpha_5)$, $A_2 = (\alpha_3\alpha_6 + \alpha_2\alpha_4)$ and $A_3 = (\alpha_3\alpha_5 - \alpha_4\alpha_7)$.

Solving equation (31) for μ , we get

$$\hat{\mu} = \frac{-A_2 \pm \sqrt{(A_2^2 - A_1A_3)}}{A_1} \tag{32}$$

It is obvious from equation (32) that the real values of μ exist if $(A_2^2 - A_1A_3) \geq 0$.

For any combination of correlations ρ_{yx} , $\rho_{y_1z_1}$, $\rho_{y_2z_2}$ and $\rho_{z_1z_2}$ which satisfy the condition of real solution, two real values of $\hat{\mu}$ are possible. Hence, while selecting the values of $\hat{\mu}$, it is to be mentioned that $0 \leq \hat{\mu} \leq 1$, all other values of $\hat{\mu}$ are inadmissible. If both the values of $\hat{\mu}$ are admissible, lowest one is the best choice.

Substituting the admissible values of $\hat{\mu}$, say μ_0 , from equation (32) into equation (30), we get the minimum value of $MSE(d)_{opt}$ in (30), as

$$\min .MSE(d)_{opt} = \frac{(\alpha_4 + \mu_0\alpha_5 + \mu_0^2\alpha_6) S_y^2}{(\alpha_3 + \mu_0\alpha_7 + \mu_0^2\alpha_8) n} . \tag{33}$$

4. Efficiency Comparisons

The percent relative efficiency of the proposed estimator ‘d’ with respect to (i) \bar{y}_n , when there is no matching; (ii) $\hat{Y} = a\bar{y}_u + (1-a)\bar{y}_{dm}$, when no auxiliary information is used at any occasion , where $\bar{y}_{dm} = \bar{y}_m + \beta_{yx}(\bar{x}_n - \bar{x}_m)$, [β_{yx} is the population regression coefficient of y on x is known] for various choices of ρ_{yx} , ρ_{y_1} , ρ_{y_2} and $\rho_{z_1z_2}$. Since \bar{y}_n and \hat{Y} are unbiased estimators of the population mean \bar{Y} , following Sukhatme et al. (1984), the variance of \bar{y}_n and optimum variance of \hat{Y} are respectively given by

$$Var(\bar{y}_n) = (1-f)(S_y^2/n) \tag{34}$$

and

$$Var(\hat{Y})_{opt} = [(1/2)(1 + \sqrt{1-\rho_{yx}^2}) - f](S_y^2/n) . \tag{35}$$

For numerical illustration, we have assumed that N=2000, n=200 and various choices of correlations. Table 1 - 4 depict the optimum values of μ and percent relative efficiencies E_1 and E_2 of the proposed estimator d with respect to \bar{y}_n and \hat{Y} respectively, where

$$E_1 = \frac{Var(\bar{y}_n)}{\min .MSE(d)_{opt}} \times 100 = \frac{(1-f)(\alpha_3 + \mu_0\alpha_7 + \mu_0^2\alpha_8)}{(\alpha_4 + \mu_0\alpha_5 + \mu_0^2\alpha_6)} \times 100, \tag{36}$$

$$E_2 = \frac{Var(\hat{Y})_{opt}}{\min .MSE(d)_{opt}} \times 100 = \frac{[(1/2)(1 + \sqrt{1-\rho_{yx}^2}) - f](\alpha_3 + \mu_0\alpha_7 + \mu_0^2\alpha_8)}{(\alpha_4 + \mu_0\alpha_5 + \mu_0^2\alpha_6)} \times 100 \tag{37}$$

Table 1: Optimum values μ_0 and PREs of d with respect to \bar{y}_n and \hat{Y} .

$\rho_{z_1z_2}$			0.3			0.4			0.5			0.6			0.7			
ρ_{yx}	ρ_{yz_1}	ρ_{yz_2}	μ_0	E_1	E_2	μ_0	E_1	E_2	μ_0	E_1	E_2	μ_0	E_1	E_2	μ_0	E_1	E_2	
0.3	0.4	-0.3	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	
		-0.4	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	0.9083	232.45	226.50	0.7953	240.64	234.49
		-0.5	NA	NA	NA	0.9083	232.45	226.50	0.7953	240.64	234.49	0.6882	252.62	246.16	0.5863	270.22	263.31	
		-0.6	0.7953	240.64	234.49	0.6882	252.62	246.16	0.5863	270.22	263.31	0.4890	296.86	289.27	0.3947	339.71	331.01	
		-0.7	0.5863	270.22	263.31	0.4890	296.86	289.27	0.3947	339.71	331.01	0.3008	417.38	406.70	0.1994	604.26	588.80	
		-0.8	0.3947	339.71	331.01	0.3008	417.38	406.70	0.1994	604.26	588.80	NA	NA	NA	NA	NA	NA	
	0.5	-0.3	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	0.9083	232.45	226.50	0.7953	240.64	234.49
		-0.4	NA	NA	NA	0.9083	232.45	226.50	0.7953	240.64	234.49	0.6882	252.62	246.16	0.5863	270.22	263.31	
		-0.5	0.7953	240.64	234.49	0.6882	252.62	246.16	0.5863	270.22	263.31	0.4890	296.86	289.27	0.3947	339.71	331.01	
		-0.6	0.5863	270.22	263.31	0.4890	296.86	289.27	0.3947	339.71	331.01	0.3008	417.38	406.70	0.1994	604.26	588.80	
		-0.7	0.3947	339.71	331.01	0.3008	417.38	406.70	0.1994	604.26	588.80	NA	NA	NA	NA	NA	NA	
		-0.8	0.1994	604.26	588.80	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	
	0.6	-0.3	NA	NA	NA	0.9083	232.45	226.50	0.7953	240.64	234.49	0.6882	252.62	246.16	0.5863	270.22	263.31	
		-0.4	0.7953	240.64	234.49	0.6882	252.62	246.16	0.5863	270.22	263.31	0.4890	296.86	289.27	0.3947	339.71	331.01	
		-0.5	0.5863	270.22	263.31	0.4890	296.86	289.27	0.3947	339.71	331.01	0.3008	417.38	406.70	0.1994	604.26	588.80	
		-0.6	0.3947	339.71	331.01	0.3008	417.38	406.70	0.1994	604.26	588.80	NA	NA	NA	NA	NA	NA	
		-0.7	0.1994	604.26	588.80	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	
		-0.8	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	
	0.7	-0.3	0.7953	240.64	234.49	0.6882	252.62	246.16	0.5863	270.22	263.31	0.4890	296.86	289.27	0.3947	339.71	331.01	
		-0.4	0.5863	270.22	263.31	0.4890	296.86	289.27	0.3947	339.71	331.01	0.3008	417.38	406.70	0.1994	604.26	588.80	
		-0.5	0.3947	339.71	331.01	0.3008	417.38	406.70	0.1994	604.26	588.80	NA	NA	NA	NA	NA	NA	
		-0.6	0.1994	604.26	588.80	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	
		-0.7	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	
		-0.8	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	

Note: 'NA' stands for 'not applicable'.

Table 2: Optimum values μ_0 and PREs of d with respect to \bar{y}_n and \hat{Y} .

$\rho_{z_1z_2}$			0.3			0.4			0.5			0.6			0.7			
ρ_{yx}	ρ_{yz_1}	ρ_{yz_2}	μ_0	E_1	E_2	μ_0	E_1	E_2	μ_0	E_1	E_2	μ_0	E_1	E_2	μ_0	E_1	E_2	
0.4	0.4	-0.3	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	
		-0.4	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	0.9940	248.90	237.35	0.8666	257.28	245.35
		-0.5	NA	NA	NA	0.9940	248.90	237.35	0.8666	257.28	245.35	0.7477	269.92	257.40	0.6359	288.73	275.34	
		-0.6	0.8666	257.28	245.35	0.7477	269.92	257.40	0.6359	288.73	275.34	0.5299	317.32	302.60	0.4277	363.34	346.49	
		-0.7	0.6359	288.73	275.34	0.5299	317.32	302.60	0.4277	363.34	346.49	0.3259	446.77	426.05	0.2159	647.68	617.64	
		-0.8	0.4277	363.34	346.49	0.3259	446.77	426.05	0.2159	647.68	617.64	NA	NA	NA	NA	NA	NA	
	0.5	-0.3	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	0.9940	248.90	237.35	0.8666	257.28	245.35
		-0.4	NA	NA	NA	0.9940	248.90	237.35	0.8666	257.28	245.35	0.7477	269.92	257.40	0.6359	288.73	275.34	
		-0.5	0.8666	257.28	245.35	0.7477	269.92	257.40	0.6359	288.73	275.34	0.5299	317.32	302.60	0.4277	363.34	346.49	
		-0.6	0.6359	288.73	275.34	0.5299	317.32	302.60	0.4277	363.34	346.49	0.3259	446.77	426.05	0.2159	647.68	617.64	
		-0.7	0.4277	363.34	346.49	0.3259	446.77	426.05	0.2159	647.68	617.64	NA	NA	NA	NA	NA	NA	
		-0.8	0.2159	647.68	617.64	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA
	0.6	-0.3	NA	NA	NA	0.9940	248.90	237.35	0.8666	257.28	245.35	0.7477	269.92	257.40	0.6359	288.73	275.34	
		-0.4	0.8666	257.28	245.35	0.7477	269.92	257.40	0.6359	288.73	275.34	0.5299	317.32	302.60	0.4277	363.34	346.49	
		-0.5	0.6359	288.73	275.34	0.5299	317.32	302.60	0.4277	363.34	346.49	0.3259	446.77	426.05	0.2159	647.68	617.64	
		-0.6	0.4277	363.34	346.49	0.3259	446.77	426.05	0.2159	647.68	617.64	NA	NA	NA	NA	NA	NA	
		-0.7	0.2159	647.68	617.64	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	
		-0.8	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	
	0.7	-0.3	0.8666	257.28	245.35	0.7477	269.92	257.40	0.6359	288.73	275.34	0.5299	317.32	302.60	0.4277	363.34	346.49	
		-0.4	0.6359	288.73	275.34	0.5299	317.32	302.60	0.4277	363.34	346.49	0.3259	446.77	426.05	0.2159	647.68	617.64	
		-0.5	0.4277	363.34	346.49	0.3259	446.77	426.05	0.2159	647.68	617.64	NA	NA	NA	NA	NA	NA	
		-0.6	0.2159	647.68	617.64	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	
		-0.7	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	
		-0.8	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	

Note: 'NA' stands for 'not applicable'.

Table 3: Optimum values μ_0 and PREs of d with respect to \bar{y}_n and \hat{Y} .

$\rho_{z_1z_2}$			0.3			0.4			0.5			0.6			0.7			
ρ_{yx}	ρ_{yz_1}	ρ_{yz_2}	μ_0	E_1	E_2	μ_0	E_1	E_2	μ_0	E_1	E_2	μ_0	E_1	E_2	μ_0	E_1	E_2	
0.5	0.4	-0.3	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	
		-0.4	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	0.9898	283.62	262.51
		-0.5	NA	NA	NA	NA	NA	NA	NA	0.9898	283.62	262.51	0.8455	295.78	273.76	0.7135	315.07	291.62
		-0.6	0.9898	283.62	262.51	0.8455	295.78	273.76	0.7135	315.07	291.62	0.5910	345.27	319.57	0.4748	394.61	365.24	
		-0.7	0.7135	315.07	291.62	0.5910	345.27	319.57	0.4748	394.61	365.24	0.3604	484.77	448.69	0.2378	703.06	650.73	
		-0.8	0.4748	394.61	365.24	0.3604	484.77	448.69	0.2378	703.06	650.73	NA	NA	NA	NA	NA	NA	
	0.5	-0.3	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	0.9898	283.62	262.51
		-0.4	NA	NA	NA	NA	NA	NA	NA	0.9898	283.62	262.51	0.8455	295.78	273.76	0.7135	315.07	291.62
		-0.5	0.9898	283.62	262.51	0.8455	295.78	273.76	0.7135	315.07	291.62	0.5910	345.27	319.57	0.4748	394.61	365.24	
		-0.6	0.7135	315.07	291.62	0.5910	345.27	319.57	0.4748	394.61	365.24	0.3604	484.77	448.69	0.2378	703.06	650.73	
		-0.7	0.4748	394.61	365.24	0.3604	484.77	448.69	0.2378	703.06	650.73	NA	NA	NA	NA	NA	NA	
		-0.8	0.2378	703.06	650.73	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	
	0.6	-0.3	NA	NA	NA	NA	NA	NA	NA	0.9898	283.62	262.51	0.8455	295.78	273.76	0.7135	315.07	291.62
		-0.4	0.9898	283.62	262.51	0.8455	295.78	273.76	0.7135	315.07	291.62	0.5910	345.27	319.57	0.4748	394.61	365.24	
		-0.5	0.7135	315.07	291.62	0.5910	345.27	319.57	0.4748	394.61	365.24	0.3604	484.77	448.69	0.2378	703.06	650.73	
		-0.6	0.4748	394.61	365.24	0.3604	484.77	448.69	0.2378	703.06	650.73	NA	NA	NA	NA	NA	NA	
		-0.7	0.2378	703.06	650.73	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	
		-0.8	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	
	0.7	-0.3	0.9898	283.62	262.51	0.8455	295.78	273.76	0.7135	315.07	291.62	0.5910	345.27	319.57	0.4748	394.61	365.24	
		-0.4	0.7135	315.07	291.62	0.5910	345.27	319.57	0.4748	394.61	365.24	0.3604	484.77	448.69	0.2378	703.06	650.73	
		-0.5	0.4748	394.61	365.24	0.3604	484.77	448.69	0.2378	703.06	650.73	NA	NA	NA	NA	NA	NA	
		-0.6	0.2378	703.06	650.73	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	
		-0.7	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	
		-0.8	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	

Note: 'NA' stands for 'not applicable'.

Table 4: Optimum values μ_0 and PREs of d with respect to \bar{y}_n and \hat{Y} .

$\rho_{z_1z_2}$			0.3			0.4			0.5			0.6			0.7				
ρ_{yx}	ρ_{yz_1}	ρ_{yz_2}	μ_0	E_1	E_2	μ_0	E_1	E_2	μ_0	E_1	E_2	μ_0	E_1	E_2	μ_0	E_1	E_2		
0.6	0.4	-0.3	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA		
		-0.4	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	
		-0.5	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	0.8475	356.36	316.77	
		-0.6	NA	NA	NA	NA	NA	NA	NA	0.8475	356.36	316.77	0.6907	386.72	343.75	0.5476	438.95	390.18	
		-0.7	0.8475	356.36	316.77	0.6907	386.72	343.75	0.5476	438.95	390.18	0.4111	536.88	477.23	0.2685	777.41	691.03		
		-0.8	0.5476	438.95	390.18	0.4111	536.88	477.23	0.2685	777.41	691.03	NA	NA	NA	NA	NA	NA	NA	
	0.5	-0.3	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	
		-0.4	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	0.8475	356.36	316.77	
		-0.5	NA	NA	NA	NA	NA	NA	NA	0.8475	356.36	316.77	0.6907	386.72	343.75	0.5476	438.95	390.18	
		-0.6	0.8475	356.36	316.77	0.6907	386.72	343.75	0.5476	438.95	390.18	0.4111	536.88	477.23	0.2685	777.41	691.03		
		-0.7	0.5476	438.95	390.18	0.4111	536.88	477.23	0.2685	777.41	691.03	NA	NA	NA	NA	NA	NA	NA	
		-0.8	0.2685	777.41	691.03	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA
	0.6	-0.3	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	0.8475	356.36	316.77	
		-0.4	NA	NA	NA	NA	NA	NA	NA	0.8475	356.36	316.77	0.6907	386.72	343.75	0.5476	438.95	390.18	
		-0.5	0.8475	356.36	316.77	0.6907	386.72	343.75	0.5476	438.95	390.18	0.4111	536.88	477.23	0.2685	777.41	691.03		
		-0.6	0.5476	438.95	390.18	0.4111	536.88	477.23	0.2685	777.41	691.03	NA	NA	NA	NA	NA	NA	NA	
		-0.7	0.2685	777.41	691.03	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA
		-0.8	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA
	0.7	-0.3	NA	NA	NA	NA	NA	NA	NA	0.8475	356.36	316.77	0.6907	386.72	343.75	0.5476	438.95	390.18	
		-0.4	0.8475	356.36	316.77	0.6907	386.72	343.75	0.5476	438.95	390.18	0.4111	536.88	477.23	0.2685	777.41	691.03		
		-0.5	0.5476	438.95	390.18	0.4111	536.88	477.23	0.2685	777.41	691.03	NA	NA	NA	NA	NA	NA	NA	
		-0.6	0.2685	777.41	691.03	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA
		-0.7	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA
		-0.8	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA

Note: 'NA' stands for 'not applicable'.

From Table 1 to 4, it is observed that

- (i) For fixed values of $(\rho_{yx}, \rho_{yz_1}, \rho_{z_1z_2})$, the values of μ_0 decrease as value of ρ_{yz_2} decreases while the values of E_1 and E_2 increase. Similar trends for (μ_0, E_1, E_2) are observed for fixed values of $(\rho_{yx}, \rho_{yz_2}, \rho_{z_1z_2})$ with increasing value of ρ_{yz_1} and also for fixed values of $(\rho_{yx}, \rho_{yz_1}, \rho_{z_1z_2})$ with increasing value of $\rho_{z_1z_2}$.
- (ii) For fixed values of $(\rho_{yz_1}, \rho_{yz_2}, \rho_{z_1z_2})$, the values of μ_0, E_1 and E_2 are increasing with increasing values of ρ_{yx} . This behavior is in agreement with Sukhatme et al. (1984), results which explained that more the value of ρ_{yx} , more the fractions of fresh sample required at the current occasion.
- (iii) Minimum value of μ_0 is 0.1994 ($\cong 0.20$), which shows that the fraction to be replaced at the current occasion is as low as about 20 percent of the total sample size leading to a reduction of considerable amount in the cost of the survey.

It is further observed from Table 1 to 4 that there is appreciable gain in efficiency by using the proposed estimator 'd' over usual unbiased estimator \bar{y}_n and the estimator \hat{Y} . Thus we conclude that the use of auxiliary information at the estimation stage is highly rewarding in terms of the proposed estimator 'd'.

5. Conclusion

In this article, we extend the current literature in two-occasion successive sampling using two auxiliary variables on both of the occasions and the information on study variable from the previous occasions one of which is positively correlated with the study variable while the other is negatively correlated. An efficient estimation procedure has been developed. Optimum replacement policy and the efficiency of the suggested estimator have been discussed. From the numerical study, it may be concluded that the proposed estimator is more beneficial in estimation of the population mean of the study variable at the current occasion in two occasion successive sampling.

Finally, looking on the nice performance of the envisaged estimator, our recommendation is to use the proposed estimator by the survey practitioners in practice.

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APPENDIX-I

- **Proof of the theorem 3.3:**

We have

$$d = \phi d_u + (1 - \phi) d_m \quad (I.1)$$

Subtracting \bar{Y} from both sides of (I.1) we have

$$d - \bar{Y} = \phi d_u + (1 - \phi) d_m - \bar{Y}$$

or

$$d - \bar{Y} = \phi (d_u - \bar{Y}) + (1 - \phi) (d_m - \bar{Y}) \quad (I.2)$$

Taking expectation of both sides of (I.2) we get the bias of d as

$$B(d) = \phi B(d_u) + (1 - \phi) B(d_m)$$

This proves the Theorem 3.3.

- **Proof of the theorem 3.4:**

Squaring both sides of (I.2) we have

$$(d - \bar{Y})^2 = \phi^2 (d_u - \bar{Y})^2 + (1 - \phi)^2 (d_m - \bar{Y})^2 + 2\phi(1 - \phi)(d_u - \bar{Y})(d_m - \bar{Y}) \quad (I.3)$$

Taking expectation of both sides of (I.3) we get the mean squared error of d as

$$MSE(d) = \phi^2 MSE(d_u) + (1 - \phi)^2 MSE(d_m) + 2\phi(1 - \phi)Cov(d_u, d_m)$$

This proves the Theorem 3.4.