Generalized Modified Ratio Type Estimator for Estimation of Population Variance

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Received: 09, May 2014/ Revised: 06, March 2015/ Accepted: 22, March 2015
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ABSTRACT

In this paper a generalized modified ratio type estimator for estimation of population variance of the study variable using the known parameters of the auxiliary variable has been proposed. The bias and mean squared error of the proposed estimators are derived. It has been shown that the ratio type variance estimator and existing modified ratio type variance estimators are the particular cases of the proposed estimators. Further the proposed estimators have been compared with that of the existing (competing) estimators for simulated data and two natural populations

Keywords: Auxiliary variable, Bias, Coefficient of Variation, Kurtosis, Mean squared error, Median, Simple random sampling, Skewness

1. Introduction

1.1 Introduction to the Research Problem

When there is no auxiliary information available, the simplest estimator of population variance is the sample variance obtained by using simple random sampling without replacement (SRSWOR). Sometimes in sample surveys, along with the study variable **Y**, information on auxiliary variable **X**, which is positively correlated with **Y**, is also available. This information on auxiliary variable **X**, may be utilized to obtain a more efficient estimator of the population variance. Ratio method of estimation is an attempt in this direction. This method of estimation may be used when (i) **X** represents the same character as **Y**, but measured at some previous date when a complete count of the population was made and (ii) Any other character **X** which is closely related to the study variable **Y** and it is cheaply, quickly and easily available (see page 77 in Gupta and Kabe (2011)).

1.2 Statement of Problem

Consider a finite population $U = \{U_1, U_2, \dots, U_N\}$ of N distinct and identifiable units. Let Y be a study variable with value Y_i measured on $U_i, i = 1,2,3,\dots,N$ giving a vector of values $Y = \{Y_1, Y_2, \dots, Y_N\}$.

The problem is to estimate the population variance $S_y^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(Y_i - \bar{Y})^2$ on the basis of a random sample of size n, selected from the population U with some desirable properties like:

- Unbiasedness / Minimum Bias
- Minimum Variance / Mean squared error

1.3 Notations

The notations to be used in this article are described below:

- $N -$ Population size
- $n -$ Sample size
- $\gamma = \frac{(1-f)}{n}$
- $Y -$ Study variable
- $X -$ Auxiliary variable
- $S_y^2, S_x^2 -$ Population variances
- $s_y^2, s_x^2 -$ Sample variances
- $C_X, C_y -$ Coefficient of variations
- $\lambda_{rs} = \frac{\mu_{rs}}{\mu_{20}^{r/2}\mu_{02}^{s/2}}$
- $\mu_{rs} = \frac{1}{N}\sum_{i=1}^{N}(Y_i - \bar{Y})^r (X_i - \bar{X})^s$
- $\beta_{1(x)} = \frac{\mu_{03}^2}{\mu_{02}^3}$, Skewness of the auxiliary variable
- $\beta_{2(x)} = \frac{\mu_{04}}{\mu_{02}^2}$, Kurtosis of the Auxiliary Variable
- $\beta_{2(y)} = \frac{\mu_{40}}{\mu_{20}^2}$, Kurtosis of the Study Variable where
- M_d −Median of the Auxiliary Variable
- Q_1 −First (lower) Quartile of the Auxiliary Variable
- Q_3 −Third (upper) Quartile of the Auxiliary Variable
- $Q_r = Q_3 - Q_1$, Inter-Quartile Range of the Auxiliary Variable

- $Q_d = \frac{Q_3 - Q_1}{2}$, Semi-Quartile Range of the Auxiliary Variable
- $Q_a = \frac{Q_3 + Q_1}{2}$, Semi-Quartile Average of the Auxiliary Variable
- $D_i -$ i^{th} Decile of the Auxiliary variable
- $B(.) -$ Bias of the estimator
- $MSE(.) -$ Mean squared error of the estimator
- $\hat{S}_R^2 -$ Ratio type variance estimator of S_y^2
- $\hat{S}_i^2 -$ Existing modified ratio type variance estimator of S_y^2
- $\hat{S}_{p_i}^2 -$ Proposed modified ratio type variance estimator of S_y^2

1.4 Simple random sampling without replacement sample variance

In the case of simple random sampling without replacement (SRSWOR), the sample variance s_y^2 is used to estimate the population variance S_y^2 which is an unbiased estimator and its variance is given below:

$$V(s_y^2) = \delta S_y^4 (\beta_{2(y)} - 1) \tag{1}$$

1.5 Ratio type estimator for estimation of population variance

Isaki (1983) suggested a ratio type variance estimator for the population variance S_y^2 when the population variance S_x^2 of the auxiliary variable X is known. The estimator together with its bias and mean squared error are given below:

$$\hat{S}_R^2 = s_y^2 \frac{S_x^2}{s_x^2} \tag{2}$$

$$B(\hat{S}_R^2) = \gamma S_y^2 \left[\left(\beta_{2(x)} - 1 \right) - (\lambda_{22} - 1) \right] \tag{3}$$

$$MSE(\hat{S}_R^2) = \gamma S_y^4 [(\beta_{2(y)} - 1) + (\beta_{2(x)} - 1) - 2(\lambda_{22} - 1)] \tag{4}$$

where $\beta_{2(y)} = \frac{\mu_{40}}{\mu_{20}^2}, \beta_{2(x)} = \frac{\mu_{04}}{\mu_{02}^2}, \lambda_{22} = \frac{\mu_{22}}{\mu_{20}\mu_{02}}$

1.6 Existing modified ratio type estimators for estimation of population variance

The ratio type variance estimator given in (2) is used to improve the precision of the estimate of the population variance compared to SRSWOR sample variance. Further improvements are also achieved on the ratio estimator by introducing a number of modified ratio estimators with the use of known parameters like Coefficient of Variation, Kurtosis, Median, Quartiles and Deciles. The problem of constructing efficient estimators for the population variance has been widely discussed by various authors such as Isaki (1983), Kadilar and Cingi (2006), Subramani and Kumarapandiyan (2012a, b, c, 2013) and Upadhyaya and Singh (1999).

Table 1 (see in Appendix A) contains all modified ratio type estimators for estimating population variance using known population parameters of the auxiliary variable in which some of the estimators are already suggested in the literature, remaining estimators have been introduced in this article. The modified ratio type estimators given in Table 1 are biased but have smaller mean squared error compared to the ratio type variance estimator suggested by Isaki (1983).

1.7 Motivations and Investigations

Moving along this direction we intend in this paper to show the problem of estimating the population variance of a study variable can be treated in a cohesive framework by defining a class of estimators which may or may not be biased and covers many that are present in the literature. The bias and mean squared error of the class are obtained. The aim is to avoid the large number of estimators that appear different from each other but, as a matter of fact, can be included in the class and therefore, their efficiency is known in advance. In this paper an attempt has been made to suggest a generalized modified ratio type estimator for estimating population variance using known parameters of the auxiliary variable and its linear combination. The materials of the present work are arranged as given below. The proposed estimators using known parameters of the auxiliary variable are presented in section 2 whereas the proposed estimators are compared theoretically with that of the SRSWOR sample variance, ratio estimator and existing modified estimators in section 3. The performance of the proposed estimators with that of the ratio and existing modified ratio estimators are assessed for certain natural populations in section 4 and the conclusion is presented in section 5

2. Generalized Modified Ratio Type Estimator

In this section, a generalized modified ratio type estimator using the known parameters of the auxiliary variable for estimating the population variance of the study variable Y has been suggested. The proposed modified ratio type estimator $\hat{S}^2_{p_i}$, $i = 1,2,\dots,51$ for estimating the population variance S^2_y is given below:

$$\hat{S}^2_{p_i} = s^2_y \left[\frac{S^2_x+\alpha\omega_i}{s^2_x+\alpha\omega_i}\right]; i = 1,2,3,\dots,51 \qquad (5)$$

The bias and mean squared error of the proposed estimators $\hat{S}^2_{p_i}$, $i = 1,2,\dots,51$ have been derived (see Appendix B) and are given below:

$$\mathrm{Bias}\left(\hat{S}_{p_i}^2\right) = \gamma S_y^2 \delta_{p_i}\left[\delta_{p_i}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right] \quad (6)$$

$$\mathrm{MSE}\left(\hat{S}_{p_i}^2\right) = \gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_{p_i}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{p_i}(\lambda_{22} - 1)\right] \quad (7)$$

where $\delta_{p_i} = \frac{S_x^2}{S_x^2 + \alpha\,\omega_i}$

Remark 2.1: When the study variable Y and auxiliary variable X are negatively correlated and the population parameters of the auxiliary variable are known, the following generalized modified product type variance estimator can be proposed:

$$\hat{S}_{pr_i}^2 = s_y^2\left[\frac{s_x^2 + \tau\,\omega_i}{S_x^2 + \tau\,\omega_i}\right]; \mathrm{i} = 1,2,3, \ldots, 51 \quad (8)$$

Remark 2.2: When $\alpha = 0$ in (5), the proposed estimator $\hat{S}_{p_i}^2$ reduces to ratio type estimator $\hat{S}_R^2$ suggested by Isaki (1983).

Remark 2.3: When $\alpha = 1$, the proposed estimator $\hat{S}_{p_i}^2$ reduces respectively to the existing estimators $\hat{S}_i^2$ listed in Table 1.

3. Efficiency of the Proposed Estimators

The mean squared error of the modified ratio type estimators $\hat{\mathbf{S}}_{\mathbf{i}}^{\mathbf{2}}; \mathbf{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots, \mathbf{51}$ given in Table 1 (Appendix A) are represented in single class as given below:

$$\mathrm{MSE}\left(\hat{S}_i^2\right) = \gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_i^2\left(\beta_{2(x)} - 1\right) - 2\delta_i(\lambda_{22} - 1)\right] \quad (9)$$

Comparing (1) and (7) we have derived (see Appendix C) the condition for which the proposed estimator $\hat{\mathbf{S}}_{\mathbf{p_i}}^{\mathbf{2}}, \mathbf{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots, \mathbf{51}$ is more efficient than the SRSWOR sample variance $\mathbf{s}_{\mathbf{y}}^{\mathbf{2}}$ and it is given below:

$$\mathrm{MSE}\left(\hat{S}_{p_i}^2\right) < V\left(s_y^2\right) \text{ if } \alpha > \omega_i^{-1} S_x^2\left[\frac{\left(\beta_{2(x)} - 1\right)}{2(\lambda_{22} - 1)} - 1\right] \quad (10)$$

Comparing (4) and (7) we have derived (see Appendix D) the conditions for which the proposed estimator $\mathbf{S}_{\mathbf{p_i}}^{\mathbf{2}}, \mathbf{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots, \mathbf{51}$ is more efficient than the ratio type estimator $\mathbf{S}_{\mathbf{R}}^{\mathbf{2}}$ and it is given below:

$$\mathrm{MSE}\left(\hat{S}_{p_i}^2\right) < \mathrm{MSE}\left(\hat{S}_R^2\right) \text{ if } 0 < \alpha < \omega_i^{-1} S_x^2\left[\frac{2\left(\beta_{2(x)} - 1\right) - 2(\lambda_{22} - 1)}{2(\lambda_{22} - 1) - \left(\beta_{2(x)} - 1\right)}\right] \quad \text{(or)}$$

$$\omega_i^{-1} S_x^2\left[\frac{2\left(\beta_{2(x)} - 1\right) - 2(\lambda_{22} - 1)}{2(\lambda_{22} - 1) - \left(\beta_{2(x)} - 1\right)}\right] < \alpha < 0 \quad (11)$$

Comparing (7) and (9) we have derived (see Appendix E) the conditions for which the proposed estimator $\mathbf{S}_{\mathbf{p_i}}^{\mathbf{2}}, \mathbf{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots, \mathbf{51}$ is more efficient than the modified ratio type variance estimator $\hat{\mathbf{S}}_{\mathbf{i}}^{\mathbf{2}}; \mathbf{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots, \mathbf{51}$ respectively and it is given below:

$$\mathrm{MSE}\left(\hat{S}_{p_i}^2\right) < \mathrm{MSE}\left(\hat{S}_i^2\right) \text{ if } 1 < \alpha < \omega_i^{-1} S_x^2 \left[\frac{(1+\delta_i)\left(\beta_{2(x)}-1\right)-2(\lambda_{22}-1)}{2(\lambda_{22}-1)-\delta_i\left(\beta_{2(x)}-1\right)}\right] \text{(or)}$$

$$\omega_i^{-1} S_x^2 \left[\frac{(1+\delta_i)\left(\beta_{2(x)}-1\right)-2(\lambda_{22}-1)}{2(\lambda_{22}-1)-\delta_i\left(\beta_{2(x)}-1\right)}\right] < \alpha < 1 \qquad (12)$$

Let us consider the lower limit point as α_L and upper limit point as α_U in (12). At the average of limit points, $\alpha_A (= \frac{\alpha_L+\alpha_U}{2})$, the proposed estimator always performs better than the existing estimators. That is,

$$\mathrm{MSE}\left(\hat{S}_{p_i}^2\right) < \mathrm{MSE}\left(\hat{S}_i^2\right) \text{ at } \alpha_A \qquad (13)$$

4. Numerical Study

The performance of the proposed modified ratio type estimators for variance are assessed with that of SRSWOR sample variance, ratio type estimator and existing modified ratio type variance estimators for two natural populations. The population 1 is taken from Singh and Chaudhary (1986, page141) and the population 2 is taken from Murthy (1967, page 228). The population parameters of the above populations are given below:

Population 1: Singh and Chaudhary (1986, page 141)

Y - Area under Lime; X - Number of bearing Lime trees

$N = 22$	$n = 5$	$\bar{Y} = 22.6$	$\bar{X} = 1467.5$	$S_y = 32.8$
$S_x = 2503.2$	$\beta_{2(x)} = 13.2$	$\beta_{1(x)} = 9.9$	$\beta_{2(y)} = 5.57$	$\lambda_{22} = 7.71$
$M_d = 534.5$	$Q_1 = 110.7$	$Q_3 = 2180.00$	$D_1 = 29.40$	$D_2 = 102.6$
$D_3 = 143.6$	$D_4 = 238.4$	$D_5 = 534.5$	$D_6 = 805.6$	$D_7 = 1499.9$
$D_8 = 2452.0$	$D_9 = 3086.3$	$D_{10} = 11799.0$		

Population 2: Murthy (1967, page 228)

Y –Output for 80 factories; X – Fixed capital for 80 factories

$N = 80$	$n = 20$	$\bar{Y} = 51.8$	$\bar{X} = 11.3$	$S_y = 18.3$
$S_x = 8.4$	$\beta_{2(x)} = 2.8$	$\beta_{1(x)} = 1.1$	$\beta_{2(y)} = 2.3$	$\lambda_{22} = 2.2$
$M_d = 7.5$	$Q_1 = 5.1$	$Q_3 = 16.9$	$D_1 = 3.6$	$D_2 = 4.6$
$D_3 = 5.9$	$D_4 = 6.7$	$D_5 = 7.5$	$D_6 = 8.5$	$D_7 = 14.8$
$D_8 = 18.1$	$D_9 = 25$	$D_{10} = 34.8$		

Variance of SRSWOR sample variance and Mean Squared Error of the ratio type estimator for the two populations are given below:

Table 2: Variance of SRSWOR Sample Variance and MSE of the Ratio Type Estimator

Estimators	MSE or Variance	
	Population 1	Population 2

SRSWOR sample variance $\mathbf{s_y^2}$	821762.3	5393.8
Ratio type estimator $\hat{\mathbf{S}}_{\mathbf{R}}^{\mathbf{2}}$	612166.8	2943.8

Further to show the efficiency of the proposed estimators (p), the Percent Relative Efficiencies (PREs) of the proposed estimators with respect to the existing estimators (e) given in Table 3 are computed by using the formula given below:

$$\text{PRE (p)} = \frac{\text{MSE(e)}}{\text{MSE(p)}} * 100$$

Table 3: PRE(.) of the proposed modified ratio type estimators

Proposed Estimators	**Popln 1**	**Popln 2**	**Proposed Estimators**	**Popln 1**	**Popln 2**
$\hat{S}^2_{p_1}$	130.4823	129.5818	$\hat{S}^2_{p_{27}}$	130.4822	130.2050
$\hat{S}^2_{p_2}$	130.4829	127.0925	$\hat{S}^2_{p_{28}}$	130.4955	119.9825
$\hat{S}^2_{p_3}$	130.4827	129.2494	$\hat{S}^2_{p_{29}}$	130.4822	130.1909
$\hat{S}^2_{p_4}$	130.4822	129.3805	$\hat{S}^2_{p_{30}}$	130.4850	121.1700
$\hat{S}^2_{p_5}$	130.6144	119.2953	$\hat{S}^2_{p_{31}}$	130.4822	130.2147
$\hat{S}^2_{p_6}$	130.5104	120.5441	$\hat{S}^2_{p_{32}}$	130.6287	118.5545
$\hat{S}^2_{p_7}$	130.4822	130.0725	$\hat{S}^2_{p_{33}}$	130.4822	130.2037
$\hat{S}^2_{p_8}$	130.4826	125.8364	$\hat{S}^2_{p_{34}}$	130.5135	119.8709
$\hat{S}^2_{p_9}$	130.4822	129.6265	$\hat{S}^2_{p_{35}}$	130.4824	129.1855
$\hat{S}^2_{p_{10}}$	130.4825	128.8585	$\hat{S}^2_{p_{36}}$	130.4822	129.4300
$\hat{S}^2_{p_{11}}$	130.4823	129.5338	$\hat{S}^2_{p_{37}}$	130.4880	123.9977
$\hat{S}^2_{p_{12}}$	130.4822	129.0388	$\hat{S}^2_{p_{38}}$	130.5973	108.8890
$\hat{S}^2_{p_{13}}$	130.4822	130.2368	$\hat{S}^2_{p_{39}}$	130.5915	114.7386
$\hat{S}^2_{p_{14}}$	130.5597	115.4650	$\hat{S}^2_{p_{40}}$	130.5368	122.9210
$\hat{S}^2_{p_{15}}$	130.4822	130.2302	$\hat{S}^2_{p_{41}}$	130.5427	115.7293
$\hat{S}^2_{p_{16}}$	130.4987	117.0272	$\hat{S}^2_{p_{42}}$	130.4837	126.0025
$\hat{S}^2_{p_{17}}$	130.4823	127.3076	$\hat{S}^2_{p_{43}}$	130.4876	124.7590
$\hat{S}^2_{p_{18}}$	130.4822	129.9666	$\hat{S}^2_{p_{44}}$	130.4898	122.8351
$\hat{S}^2_{p_{19}}$	130.4830	126.8675	$\hat{S}^2_{p_{45}}$	130.4948	121.6964
$\hat{S}^2_{p_{20}}$	130.4822	130.0062	$\hat{S}^2_{p_{46}}$	130.5104	120.5441
$\hat{S}^2_{p_{21}}$	130.4822	129.9996	$\hat{S}^2_{p_{47}}$	130.5248	119.2270
$\hat{S}^2_{p_{22}}$	130.4921	126.9882	$\hat{S}^2_{p_{48}}$	130.5614	111.1183
$\hat{S}^2_{p_{23}}$	130.4822	129.9630	$\hat{S}^2_{p_{49}}$	130.6117	107.8257

$\hat{S}^2_{p_{24}}$	130.4843	127.3785	$\hat{S}^2_{p_{50}}$	130.6452	102.9536
$\hat{S}^2_{p_{25}}$	130.4828	129.1772	$\hat{S}^2_{p_{51}}$	131.1041	100.1304
$\hat{S}^2_{p_{26}}$	130.4822	129.4395			

If PRE $(p) > 100$ implies that the proposed estimators are performing better than the existing estimators. It is to be noted that the PRE values are independent of the sample size. From the PRE values given in Table 3, it is observed that the proposed estimators are performed better than the existing estimators. In fact PRE of the proposed estimators varies from 130.48 to 131.10 for population 1 and from 100.13 to 130.24 for population 2. Hence one may conclude from the numerical comparison that the proposed estimators are more efficient than the existing estimators.

5. Simulation study

However to assess more about the efficiency of the proposed estimators, we have undertaken a simulation study as given below: We generate $N = 200$ values (x_i, y_i) from a Bi-variate normal distribution with means (50, 50) and standard deviation (10, 10). The correlation coefficient is fixed at values 0.90 and 0.95. Simple random sampling without replacement has been considered for sample size $n = 20$. Since the PREs are independent to the sample size we have restricted the simulation study for sample size 20 only. Further we have generated 1000 times a finite population of size 200 and compute various parameters and present the average values both in tabular and graphical form. For different values of correlation coefficient, the corresponding simulated population means and standard deviations are given in the following table:

ρ	$\bar{X}$	$\bar{Y}$	S_x	S_y
0.95	50.0365	49.9737	10.4820	10.4615
0.90	51.2780	51.2052	10.5836	10.6743

Variance of SRSWOR sample variance and Mean Squared Error of the ratio type estimator for simulated data are given below:

Table 4: Variance of SRSWOR sample variance and MSE of the ratio type estimator

Estimator	**MSE or Variance**	
	$\boldsymbol{\rho = 0.95}$	$\boldsymbol{\rho = 0.90}$
SRSWOR sample variance $\mathbf{s_y^2}$	1378.5990	864.8083
Ratio type estimator $\mathbf{\hat{S}_R^2}$	755.1780	303.7316

The pre of the existing and proposed modified ratio type variance estimators for different values of $n = 20$ for $\rho = 0.90$ and $\rho = 0.95$ are given in the following table:

Table 5: PRE(.) of the proposed modified ratio type estimators for the values of $n = 20$, $\rho = 0.90$ and $\rho = 0.95$

Proposed Estimators	$\rho = 0.90$	$\rho = 0.95$	Proposed Estimators	$\rho = 0.90$	$\rho = 0.95$
$\hat{S}^2_{p_1}$	119.73	100.12	$\hat{S}^2_{p_{27}}$	119.98	100.09
$\hat{S}^2_{p_2}$	116.45	101.19	$\hat{S}^2_{p_{28}}$	110.22	198.33
$\hat{S}^2_{p_3}$	119.81	100.10	$\hat{S}^2_{p_{29}}$	120.00	100.09
$\hat{S}^2_{p_4}$	118.81	100.29	$\hat{S}^2_{p_{30}}$	139.73	140.47
$\hat{S}^2_{p_5}$	108.46	107.34	$\hat{S}^2_{p_{31}}$	119.89	100.10
$\hat{S}^2_{p_6}$	103.22	167.10	$\hat{S}^2_{p_{32}}$	107.60	107.84
$\hat{S}^2_{p_7}$	119.91	100.10	$\hat{S}^2_{p_{33}}$	119.99	100.09
$\hat{S}^2_{p_8}$	106.24	113.52	$\hat{S}^2_{p_{34}}$	104.92	169.78
$\hat{S}^2_{p_9}$	118.18	100.61	$\hat{S}^2_{p_{35}}$	119.74	100.12
$\hat{S}^2_{p_{10}}$	119.06	100.19	$\hat{S}^2_{p_{36}}$	113.93	102.12
$\hat{S}^2_{p_{11}}$	119.71	100.12	$\hat{S}^2_{p_{37}}$	101.07	158.15
$\hat{S}^2_{p_{12}}$	114.56	102.00	$\hat{S}^2_{p_{38}}$	105.28	175.56
$\hat{S}^2_{p_{13}}$	119.98	100.09	$\hat{S}^2_{p_{39}}$	105.29	110.98
$\hat{S}^2_{p_{14}}$	102.86	167.08	$\hat{S}^2_{p_{40}}$	111.31	103.62
$\hat{S}^2_{p_{15}}$	120.00	100.09	$\hat{S}^2_{p_{41}}$	102.61	167.48
$\hat{S}^2_{p_{16}}$	143.30	170.98	$\hat{S}^2_{p_{42}}$	100.12	145.75
$\hat{S}^2_{p_{17}}$	103.11	136.18	$\hat{S}^2_{p_{43}}$	100.77	156.10
$\hat{S}^2_{p_{18}}$	119.93	100.10	$\hat{S}^2_{p_{44}}$	101.57	159.73
$\hat{S}^2_{p_{19}}$	116.12	101.26	$\hat{S}^2_{p_{45}}$	102.36	164.13
$\hat{S}^2_{p_{20}}$	119.57	100.14	$\hat{S}^2_{p_{46}}$	103.22	167.10
$\hat{S}^2_{p_{21}}$	119.66	100.14	$\hat{S}^2_{p_{47}}$	104.06	170.46
$\hat{S}^2_{p_{22}}$	115.25	101.21	$\hat{S}^2_{p_{48}}$	104.89	174.35
$\hat{S}^2_{p_{23}}$	119.93	100.10	$\hat{S}^2_{p_{49}}$	105.81	177.81
$\hat{S}^2_{p_{24}}$	103.13	113.74	$\hat{S}^2_{p_{50}}$	106.95	181.50
$\hat{S}^2_{p_{25}}$	119.79	100.11	$\hat{S}^2_{p_{51}}$	110.77	189.60
$\hat{S}^2_{p_{26}}$	112.63	105.61			

From the PRE values it is observed that the proposed estimators are performed better than the existing estimators. In fact PRE of the proposed estimators varies from 100.12 to 143.30 for$\rho = 0.90$ and from 100.09 to 198.33 for $\rho = 0.95$. This

shows that the proposed estimators are more efficient than the existing estimators.
In order to show the performances of the proposed estimators graphically, we have simulated 200 samples 1000 times and repeated the same procedures 10 times and shown the average MSE values of the estimators in the below tables and graphs. We have considered only three estimators namely $\hat{S}^2_{28}, \hat{S}^2_{38}$ and $\hat{S}^2_{51}$ for comparison with the proposed estimators.

Table 6: Mean squared error of the estimators for n = 20 and ρ = 0.90

Sample Number	Mean Squared Error					
	Existing estimators			Proposed estimators		
	$\hat{S}^2_{28}$	$\hat{S}^2_{38}$	$\hat{S}^2_{51}$	$\hat{S}^2_{p_{28}}$	$\hat{S}^2_{p_{38}}$	$\hat{S}^2_{p_{51}}$
1	273.91	261.36	276.00	248.86	248.96	248.98
2	273.82	261.51	275.89	248.96	249.06	249.08
3	275.31	262.08	276.32	250.31	250.41	250.43
4	273.64	261.03	276.60	248.79	248.89	248.91
5	274.23	261.26	276.08	249.09	249.19	249.22
6	274.34	260.91	275.45	249.27	249.37	249.40
7	274.15	262.50	276.13	249.18	249.28	249.31
8	274.68	262.29	276.60	249.71	249.81	249.83
9	274.01	261.04	276.12	248.73	248.83	248.85
10	274.50	261.20	276.90	249.43	249.53	249.55

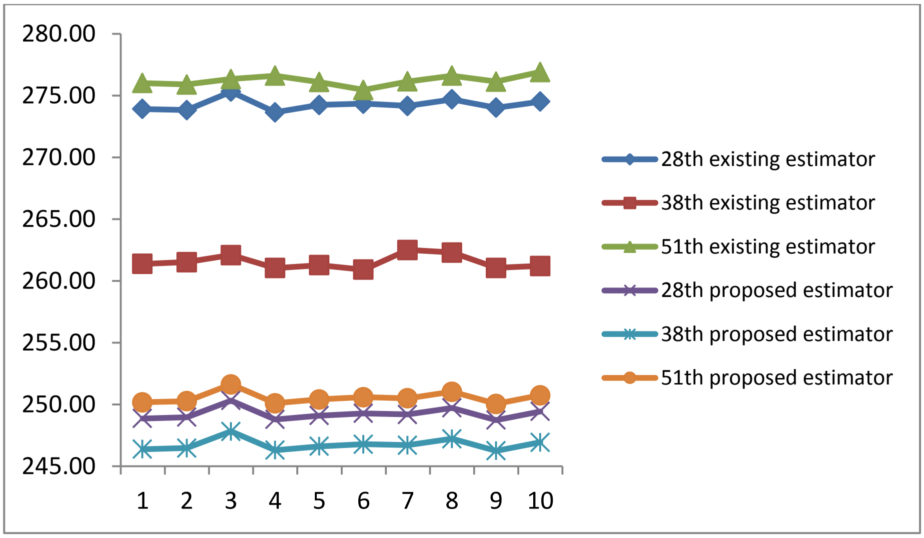

Figure 1: MSE of the estimators for n = 20 and ρ = 0.90

Table 7: Mean squared error of the estimators for $n = 20$ and $\rho = 0.95$

Sample Number	Mean Squared Error					
	Existing estimators			Proposed estimators		
	$\hat{S}^2_{28}$	$\hat{S}^2_{38}$	$\hat{S}^2_{51}$	$\hat{S}^2_{p_{28}}$	$\hat{S}^2_{p_{38}}$	$\hat{S}^2_{p_{51}}$
1	430.71	303.86	351.12	217.36	173.89	189.54
2	430.67	304.41	349.60	217.46	173.96	189.62
3	430.49	303.30	350.97	217.07	173.66	189.29
4	430.36	304.21	351.69	217.13	173.71	189.34
5	430.07	304.30	350.28	217.25	173.80	189.44
6	430.12	304.10	350.06	216.84	173.47	189.08
7	430.03	304.91	349.98	216.83	173.47	189.08
8	430.46	303.80	350.24	216.83	173.47	189.08
9	429.44	304.50	351.16	215.82	172.65	188.19
10	430.26	304.64	350.98	216.62	173.30	188.89

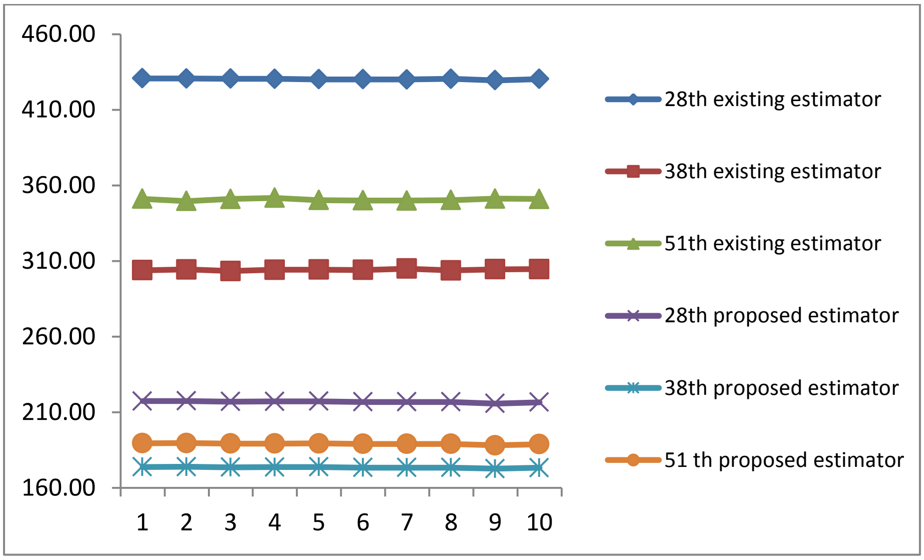

Figure 2: MSE of the estimators for $n = 20$ and $\rho = 0.95$

From the above figures 1 and 2; and tables 6 and 7, it is clear that the proposed estimators perform better than the existing estimators.

6. Conclusion

In this paper a generalized modified ratio type estimator for estimating population variance using the known parameters of the auxiliary variable has been proposed. The bias and mean squared error of the proposed modified ratio type estimators are derived. Further it has been shown that ratio and existing modified ratio type estimators are the particular cases of the proposed estimators. We have also assessed the performances of the proposed estimators with that of the existing estimators for simulated data and two natural populations. It is observed from the numerical comparison that the mean squared error of the proposed estimators is less than the mean squared error/variance of the existing (competing) estimators. Hence we strongly recommend that the proposed modified ratio type estimators for the use of practical applications for estimation of population variance.

Acknowledgements

The author is thankful to the editor and the reviewers for their constructive comments, which have improved the presentation of the paper. Further, the author wishes to record his gratitude and thanks to UGC-MRP, New Delhi, for the financial assistance.

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Appendix A

Table 1: Modified ratio type estimators for estimating population variance with the bias and mean squared error

Estimator	Bias - B(.)	Mean squared error MSE(.)
$\hat{S}_1^2 = s_y^2\left[\frac{S_x^2 + C_x}{s_x^2 + C_x}\right]$ Kadilar and Cingi (2006)	$\gamma S_y^2 \delta_1 \left[\delta_1 \left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4 \left[\left(\beta_{2(y)} - 1\right) + \delta_1^2 \left(\beta_{2(x)} - 1\right) - 2\delta_1(\lambda_{22} - 1)\right]$
$\hat{S}_2^2 = s_y^2\left[\frac{S_x^2 + \beta_{2(x)}}{s_x^2 + \beta_{2(x)}}\right]$ Upadhyaya and Singh (1999)	$\gamma S_y^2 \delta_2 \left[\delta_2 \left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4 \left[\left(\beta_{2(y)} - 1\right) + \delta_2^2 \left(\beta_{2(x)} - 1\right) - 2\delta_2(\lambda_{22} - 1)\right]$
$\hat{S}_3^2 = s_y^2\left[\frac{S_x^2 + \beta_{1(x)}}{s_x^2 + \beta_{1(x)}}\right]$	$\gamma S_y^2 \delta_3 \left[\delta_3 \left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4 \left[\left(\beta_{2(y)} - 1\right) + \delta_3^2 \left(\beta_{2(x)} - 1\right) - 2\delta_3(\lambda_{22} - 1)\right]$
$\hat{S}_4^2 = s_y^2\left[\frac{S_x^2 + \rho}{s_x^2 + \rho}\right]$	$\gamma S_y^2 \delta_4 \left[\delta_4 \left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4 \left[\left(\beta_{2(y)} - 1\right) + \delta_4^2 \left(\beta_{2(x)} - 1\right) - 2\delta_4(\lambda_{22} - 1)\right]$
$\hat{S}_5^2 = s_y^2\left[\frac{S_x^2 + S_x}{s_x^2 + S_x}\right]$	$\gamma S_y^2 \delta_5 \left[\delta_5 \left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4 \left[\left(\beta_{2(y)} - 1\right) + \delta_5^2 \left(\beta_{2(x)} - 1\right) - 2\delta_5(\lambda_{22} - 1)\right]$
$\hat{S}_6^2 = s_y^2\left[\frac{S_x^2 + M_d}{s_x^2 + M_d}\right]$ Subramani and Kumarapandiyan (2012a)	$\gamma S_y^2 \delta_6 \left[\delta_6 \left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4 \left[\left(\beta_{2(y)} - 1\right) + \delta_6^2 \left(\beta_{2(x)} - 1\right) - 2\delta_6(\lambda_{22} - 1)\right]$

$\hat{S}_7^2 = s_y^2\left[\frac{S_x^2 + Q_1}{s_x^2 + Q_1}\right]$ Subramani and Kumarapandiyan (2012b)	$\gamma S_y^2 \delta_7 \left[\delta_7 \left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4 \left[\left(\beta_{2(y)} - 1\right) + \delta_7^2 \left(\beta_{2(x)} - 1\right) - 2\delta_7(\lambda_{22} - 1)\right]$
$\hat{S}_8^2 = s_y^2\left[\frac{S_x^2 + Q_3}{s_x^2 + Q_3}\right]$ Subramani and Kumarapandiyan (2012b)	$\gamma S_y^2 \delta_8 \left[\delta_8 \left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4 \left[\left(\beta_{2(y)} - 1\right) + \delta_8^2 \left(\beta_{2(x)} - 1\right) - 2\delta_8(\lambda_{22} - 1)\right]$
$\hat{S}_9^2 = s_y^2\left[\frac{S_x^2 + Q_r}{s_x^2 + Q_r}\right]$ Subramani and Kumarapandiyan (2012b)	$\gamma S_y^2 \delta_9 \left[\delta_9 \left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4 \left[\left(\beta_{2(y)} - 1\right) + \delta_9^2 \left(\beta_{2(x)} - 1\right) - 2\delta_9(\lambda_{22} - 1)\right]$
$\hat{S}_{10}^2 = s_y^2\left[\frac{S_x^2 + Q_d}{s_x^2 + Q_d}\right]$ Subramani and Kumarapandiyan (2012b)	$\gamma S_y^2 \delta_{10} \left[\delta_{10} \left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4 \left[\left(\beta_{2(y)} - 1\right) + \delta_{10}^2 \left(\beta_{2(x)} - 1\right) - 2\delta_{10}(\lambda_{22} - 1)\right]$
$\hat{S}_{11}^2 = s_y^2\left[\frac{S_x^2 + Q_a}{s_x^2 + Q_a}\right]$ Subramani and Kumarapandiyan (2012b)	$\gamma S_y^2 \delta_{11} \left[\delta_{11} \left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4 \left[\left(\beta_{2(y)} - 1\right) + \delta_{11}^2 \left(\beta_{2(x)} - 1\right) - 2\delta_{11}(\lambda_{22} - 1)\right]$
$\hat{S}_{12}^2 = s_y^2\left[\frac{S_x^2 + D_1}{s_x^2 + D_1}\right]$ Subramani and Kumarapandiyan (2012c)	$\gamma S_y^2 \delta_{12} \left[\delta_{12} \left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4 \left[\left(\beta_{2(y)} - 1\right) + \delta_{12}^2 \left(\beta_{2(x)} - 1\right) - 2\delta_{12}(\lambda_{22} - 1)\right]$
$\hat{S}_{13}^2 = s_y^2\left[\frac{S_x^2 + D_2}{s_x^2 + D_2}\right]$ Subramani and Kumarapandiyan (2012c)	$\gamma S_y^2 \delta_{13} \left[\delta_{13} \left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4 \left[\left(\beta_{2(y)} - 1\right) + \delta_{13}^2 \left(\beta_{2(x)} - 1\right) - 2\delta_{13}(\lambda_{22} - 1)\right]$
$\hat{S}_{14}^2 = s_y^2\left[\frac{S_x^2 + D_3}{s_x^2 + D_3}\right]$ Subramani and Kumarapandiyan (2012c)	$\gamma S_y^2 \delta_{14} \left[\delta_{14} \left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4 \left[\left(\beta_{2(y)} - 1\right) + \delta_{14}^2 \left(\beta_{2(x)} - 1\right) - 2\delta_{14}(\lambda_{22} - 1)\right]$
$\hat{S}_{15}^2 = s_y^2\left[\frac{S_x^2 + D_4}{s_x^2 + D_4}\right]$ Subramani and Kumarapandiyan (2012c)	$\gamma S_y^2 \delta_{15} \left[\delta_{15} \left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4 \left[\left(\beta_{2(y)} - 1\right) + \delta_{15}^2 \left(\beta_{2(x)} - 1\right) - 2\delta_{15}(\lambda_{22} - 1)\right]$
$\hat{S}_{16}^2 = s_y^2\left[\frac{S_x^2 + D_5}{s_x^2 + D_5}\right]$ Subramani and Kumarapandiyan (2012c)	$\gamma S_y^2 \delta_{16} \left[\delta_{16} \left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4 \left[\left(\beta_{2(y)} - 1\right) + \delta_{16}^2 \left(\beta_{2(x)} - 1\right) - 2\delta_{16}(\lambda_{22} - 1)\right]$
$\hat{S}_{17}^2 = s_y^2\left[\frac{S_x^2 + D_6}{s_x^2 + D_6}\right]$ Subramani and Kumarapandiyan (2012c)	$\gamma S_y^2 \delta_{17} \left[\delta_{17} \left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4 \left[\left(\beta_{2(y)} - 1\right) + \delta_{17}^2 \left(\beta_{2(x)} - 1\right) - 2\delta_{17}(\lambda_{22} - 1)\right]$
$\hat{S}_{18}^2 = s_y^2\left[\frac{S_x^2 + D_7}{s_x^2 + D_7}\right]$ Subramani and Kumarapandiyan (2012c)	$\gamma S_y^2 \delta_{18} \left[\delta_{18} \left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4 \left[\left(\beta_{2(y)} - 1\right) + \delta_{18}^2 \left(\beta_{2(x)} - 1\right) - 2\delta_{18}(\lambda_{22} - 1)\right]$
$\hat{S}_{19}^2 = s_y^2\left[\frac{S_x^2 + D_8}{s_x^2 + D_8}\right]$ Subramani and Kumarapandiyan (2012c)	$\gamma S_y^2 \delta_{19} \left[\delta_{19} \left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4 \left[\left(\beta_{2(y)} - 1\right) + \delta_{19}^2 \left(\beta_{2(x)} - 1\right) - 2\delta_{1\circ}(\lambda_{22} - 1)\right]$

$\hat{S}_{20}^2 = s_y^2\left[\frac{S_x^2 + D_9}{s_x^2 + D_9}\right]$ Subramani and Kumarapandiyan (2012c)	$\gamma S_y^2 \delta_{20}\left[\delta_{20}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_{20}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{20}(\lambda_{22} - 1)\right]$
$\hat{S}_{21}^2 = s_y^2\left[\frac{S_x^2 + D_{10}}{s_x^2 + D_{10}}\right]$ Subramani and Kumarapandiyan (2012c)	$\gamma S_y^2 \delta_{21}\left[\delta_{21}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_{21}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{21}(\lambda_{22} - 1)\right]$
$\hat{S}_{22}^2 = s_y^2\left[\frac{\beta_{2(x)}S_x^2 + C_x}{\beta_{2(x)}s_x^2 + C_x}\right]$ Kadilar and Cingi (2006)	$\gamma S_y^2 \delta_{22}\left[\delta_{22}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_{22}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{22}(\lambda_{22} - 1)\right]$
$\hat{S}_{23}^2 = s_y^2\left[\frac{C_xS_x^2 + \beta_{2(x)}}{C_xs_x^2 + \beta_{2(x)}}\right]$ Kadilar and Cingi (2006)	$\gamma S_y^2 \delta_{23}\left[\delta_{23}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_{23}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{23}(\lambda_{22} - 1)\right]$
$\hat{S}_{24}^2 = s_y^2\left[\frac{\beta_{1(x)}S_x^2 + C_x}{\beta_{1(x)}s_x^2 + C_x}\right]$	$\gamma S_y^2 \delta_{24}\left[\delta_{24}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_{24}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{24}(\lambda_{22} - 1)\right]$
$\hat{S}_{25}^2 = s_y^2\left[\frac{C_xS_x^2 + \beta_{1(x)}}{C_xs_x^2 + \beta_{1(x)}}\right]$	$\gamma S_y^2 \delta_{25}\left[\delta_{25}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_{25}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{25}(\lambda_{22} - 1)\right]$
$\hat{S}_{26}^2 = s_y^2\left[\frac{\rho S_x^2 + C_x}{\rho s_x^2 + C_x}\right]$	$\gamma S_y^2 \delta_{26}\left[\delta_{26}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_{26}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{26}(\lambda_{22} - 1)\right]$
$\hat{S}_{27}^2 = s_y^2\left[\frac{C_xS_x^2 + \rho}{C_xs_x^2 + \rho}\right]$	$\gamma S_y^2 \delta_{27}\left[\delta_{27}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_{27}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{27}(\lambda_{22} - 1)\right]$
$\hat{S}_{28}^2 = s_y^2\left[\frac{S_xS_x^2 + C_x}{S_xs_x^2 + C_x}\right]$	$\gamma S_y^2 \delta_{28}\left[\delta_{28}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_{28}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{28}(\lambda_{22} - 1)\right]$
$\hat{S}_{29}^2 = s_y^2\left[\frac{C_xS_x^2 + S_x}{C_xs_x^2 + S_x}\right]$	$\gamma S_y^2 \delta_{29}\left[\delta_{29}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_{29}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{29}(\lambda_{22} - 1)\right]$
$\hat{S}_{30}^2 = s_y^2\left[\frac{M_dS_x^2 + C_x}{M_ds_x^2 + C_x}\right]$	$\gamma S_y^2 \delta_{30}\left[\delta_{30}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_{30}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{30}(\lambda_{22} - 1)\right]$
$\hat{S}_{31}^2 = s_y^2\left[\frac{C_xS_x^2 + M_d}{C_xs_x^2 + M_d}\right]$ Subramani and Kumarapandiyan (2013)	$\gamma S_y^2 \delta_{31}\left[\delta_{31}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_{31}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{31}(\lambda_{22} - 1)\right]$
$\hat{S}_{32}^2 = s_y^2\left[\frac{\beta_{1(x)}S_x^2 + \beta_{2(x)}}{\beta_{1(x)}s_x^2 + \beta_{2(x)}}\right]$	$\gamma S_y^2 \delta_{32}\left[\delta_{32}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_{32}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{32}(\lambda_{22} - 1)\right]$
$\hat{S}_{33}^2 = s_y^2\left[\frac{\beta_{2(x)}S_x^2 + \beta_{1(x)}}{\beta_{2(x)}s_x^2 + \beta_{1(x)}}\right]$	$\gamma S_y^2 \delta_{33}\left[\delta_{33}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_{33}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{33}(\lambda_{22} - 1)\right]$
$\hat{S}_{34}^2 = s_y^2\left[\frac{\rho S_x^2 + \beta_{2(x)}}{\rho s_x^2 + \beta_{2(x)}}\right]$	$\gamma S_y^2 \delta_{34}\left[\delta_{34}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_{34}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{34}(\lambda_{22} - 1)\right]$
$\hat{S}_{35}^2 = s_y^2\left[\frac{\beta_{2(x)}S_x^2 + \rho}{\beta_{2(x)}s_x^2 + \rho}\right]$	$\gamma S_y^2 \delta_{35}\left[\delta_{35}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_{35}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{35}(\lambda_{22} - 1)\right]$
$\hat{S}_{36}^2 = s_y^2\left[\frac{S_xS_x^2 + \beta_{2(x)}}{S_xs_x^2 + \beta_{2(x)}}\right]$	$\gamma S_y^2 \delta_{36}\left[\delta_{36}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_{36}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{36}(\lambda_{22} - 1)\right]$
$\hat{S}_{37}^2 = s_y^2\left[\frac{\beta_{2(x)}S_x^2 + S_x}{\beta_{2(x)}s_x^2 + S_x}\right]$	$\gamma S_y^2 \delta_{37}\left[\delta_{37}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_{37}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{37}(\lambda_{22} - 1)\right]$
$\hat{S}_{38}^2 = s_y^2\left[\frac{M_dS_x^2 + \beta_{2(x)}}{M_ds_x^2 + \beta_{2(x)}}\right]$	$\gamma S_y^2 \delta_{38}\left[\delta_{38}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_{38}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{38}(\lambda_{22} - 1)\right]$

$\hat{S}^2_{39} = s^2_y\left[\frac{\beta_{2(x)}S^2_x + M_d}{\beta_{2(x)}s^2_x + M_d}\right]$	$\gamma S^2_y\delta_{39}\left[\delta_{39}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{39}\left(\beta_{2(x)} - 1\right) - 2\delta_{39}(\lambda_{22} - 1)\right]$
$\hat{S}^2_{40} = s^2_y\left[\frac{\rho S^2_x + \beta_{1(x)}}{\rho s^2_x + \beta_{1(x)}}\right]$	$\gamma S^2_y\delta_{40}\left[\delta_{40}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{40}\left(\beta_{2(x)} - 1\right) - 2\delta_{40}(\lambda_{22} - 1)\right]$
$\hat{S}^2_{41} = s^2_y\left[\frac{\beta_{1(x)}S^2_x + \rho}{\beta_{1(x)}s^2_x + \rho}\right]$	$\gamma S^2_y\delta_{41}\left[\delta_{41}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{41}\left(\beta_{2(x)} - 1\right) - 2\delta_{41}(\lambda_{22} - 1)\right]$
$\hat{S}^2_{42} = s^2_y\left[\frac{S_xS^2_x + \beta_{1(x)}}{S_xs^2_x + \beta_{1(x)}}\right]$	$\gamma S^2_y\delta_{42}\left[\delta_{42}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{42}\left(\beta_{2(x)} - 1\right) - 2\delta_{42}(\lambda_{22} - 1)\right]$
$\hat{S}^2_{43} = s^2_y\left[\frac{\beta_{1(x)}S^2_x + S_x}{\beta_{1(x)}s^2_x + S_x}\right]$	$\gamma S^2_y\delta_{43}\left[\delta_{43}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{43}\left(\beta_{2(x)} - 1\right) - 2\delta_{43}(\lambda_{22} - 1)\right]$
$\hat{S}^2_{44} = s^2_y\left[\frac{M_dS^2_x + \beta_{1(x)}}{M_ds^2_x + \beta_{1(x)}}\right]$	$\gamma S^2_y\delta_{44}\left[\delta_{44}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{44}\left(\beta_{2(x)} - 1\right) - 2\delta_{44}(\lambda_{22} - 1)\right]$
$\hat{S}^2_{45} = s^2_y\left[\frac{\beta_{1(x)}S^2_x + M_d}{\beta_{1(x)}s^2_x + M_d}\right]$	$\gamma S^2_y\delta_{45}\left[\delta_{45}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{45}\left(\beta_{2(x)} - 1\right) - 2\delta_{45}(\lambda_{22} - 1)\right]$
$\hat{S}^2_{46} = s^2_y\left[\frac{S_xS^2_x + \rho}{S_xs^2_x + \rho}\right]$	$\gamma S^2_y\delta_{46}\left[\delta_{46}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{46}\left(\beta_{2(x)} - 1\right) - 2\delta_{46}(\lambda_{22} - 1)\right]$
$\hat{S}^2_{47} = s^2_y\left[\frac{\rho S^2_x + S_x}{\rho s^2_x + S_x}\right]$	$\gamma S^2_y\delta_{47}\left[\delta_{47}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{47}\left(\beta_{2(x)} - 1\right) - 2\delta_{47}(\lambda_{22} - 1)\right]$
$\hat{S}^2_{48} = s^2_y\left[\frac{M_dS^2_x + \rho}{S_xs^2_x + \rho}\right]$	$\gamma S^2_y\delta_{48}\left[\delta_{48}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{48}\left(\beta_{2(x)} - 1\right) - 2\delta_{48}(\lambda_{22} - 1)\right]$
$\hat{S}^2_{49} = s^2_y\left[\frac{\rho S^2_x + M_d}{\rho s^2_x + M_d}\right]$	$\gamma S^2_y\delta_{49}\left[\delta_{49}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{49}\left(\beta_{2(x)} - 1\right) - 2\delta_{49}(\lambda_{22} - 1)\right]$
$\hat{S}^2_{50} = s^2_y\left[\frac{M_dS^2_x + S_x}{M_ds^2_x + S_x}\right]$	$\gamma S^2_y\delta_{50}\left[\delta_{50}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{50}\left(\beta_{2(x)} - 1\right) - 2\delta_{50}(\lambda_{22} - 1)\right]$
$\hat{S}^2_{51} = s^2_y\left[\frac{S_xS^2_x + M_d}{S_xs^2_x + M_d}\right]$	$\gamma S^2_y\delta_{51}\left[\delta_{51}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{51}\left(\beta_{2(x)} - 1\right) - 2\delta_{51}(\lambda_{22} - 1)\right]$

where$\delta_i = \frac{S^2_x}{S^2_x+\omega_i}; i = 1,2,3,\ldots,51; \omega_1 = C_x, \omega_2 = \beta_{2(x)}, \omega_3 = \beta_{1(x)}, \omega_4 = \rho, \omega_6 = M_d,$
$\omega_7 = Q_1, \omega_8 = Q_3, \omega_9 = Q_r, \omega_{10} = Q_d, \omega_{11} = Q_a, \omega_{12} = D_1, \omega_{14} = D_3, \omega_{15} = D_4,$
$\omega_{16} = D_5, \omega_{17} = D_6, \omega_{18} = D_7, \omega_{19} = D_8, \omega_{20} = D_9, \omega_{21} = D_{10}, \omega_{22} = \frac{C_x}{\beta_{2(x)}},$
$\omega_{23=} = \frac{\beta_{2(x)}}{C_x}, \omega_{24} = \frac{C_x}{\beta_{1(x)}}, \omega_{25} = \frac{\beta_{1(x)}}{C_x}, \omega_{26} = \frac{C_x}{\rho}, \omega_{27} = \frac{\rho}{C_x}, \omega_{28} = \frac{C_x}{S_x}, \omega_{29} = \frac{S_x}{C_x},$
$\omega_{30} = \frac{C_x}{M_d}, \omega_{31} = \frac{M_d}{C_x}, \omega_{32} = \frac{\beta_{2(x)}}{\beta_{1(x)}}, \omega_{33} = \frac{\beta_{1(x)}}{\beta_{2(x)}}, \omega_{34} = \frac{\beta_{2(x)}}{\rho}, \omega_{35} = \frac{\rho}{\beta_{2(x)}}, \omega_{37} = \frac{S_x}{\beta_{2(x)}}$
$\omega_{38} = \frac{\beta_{2(x)}}{M_d}, \omega_{39} = \frac{M_d}{\beta_{2(x)}}, \ \omega_{40} = \frac{\beta_{1(x)}}{\rho}, \omega_{41} = \frac{\rho}{\beta_{1(x)}}, \omega_{42} = \frac{\beta_{1(x)}}{S_x}, \omega_{43} = \frac{S_x}{\beta_{1(x)}},$
$\omega_{44} = \frac{\beta_{1(x)}}{M_d}, \omega_{45} = \frac{M_d}{\beta_{1(x)}}, \omega_{46} = \frac{\rho}{S_x}, \omega_{47} = \frac{S_x}{\rho}, \omega_{48} = \frac{\beta_{1(x)}}{M_d}, \omega_{49} = \frac{M_d}{\beta_{1(x)}}, \omega_{50} = \frac{S_x}{M_d}$ and
$\omega_{51} = \frac{M_d}{S_x}$

Appendix B

We have derived here the bias and MSE of the proposed estimator $\hat{S}^2_{p_i}: i = 1,2,3,\dots,51$ to first order of approximation as given below:

Let $e_0 = \frac{s_y^2 - S_y^2}{S_y^2}$ and $e_1 = \frac{s_x^2 - S_x^2}{S_x^2}$. Further we can write $s_y^2 = S_y^2(1+e_0)$ and $s_x^2 = S_x^2(1+e_0)$ and from the definition of e_0 and e_1 we obtain:

$$E[e_0] = E[e_1] = 0$$

$$E[e_0^2] = \frac{(1-f)}{n}\left(\beta_{2(y)} - 1\right)$$

$$E[e_1^2] = \frac{(1-f)}{n}\left(\beta_{2(x)} - 1\right)$$

$$E[e_0 e_1] = \frac{(1-f)}{n}(\lambda_{22} - 1)$$

The proposed estimator $\hat{S}^2_{p_i}; i = 1,2,3,\dots,51$ is given below:

$$\hat{S}^2_{p_i} = s_y^2\left[\frac{S_x^2 + \alpha\omega_i}{s_x^2 + \alpha\omega_i}\right]$$

$$\Rightarrow \hat{S}^2_{p_i} = S_y^2(1+e_0)\left[\frac{S_x^2 + \alpha\omega_i}{(S_x^2 + e_1 S_x^2 + \alpha\omega_i)}\right]$$

$$\Rightarrow \hat{S}^2_{p_i} = \frac{S_y^2(1+e_0)}{\left(1+\delta_{p_i}e_1\right)} \text{ where } \delta_{p_i} = \frac{S_x^2}{S_x^2 + \alpha\omega_i}$$

$$\Rightarrow \hat{S}^2_{p_i} = S_y^2(1+e_0)\left(1+\delta_{p_i}e_1\right)^{-1}$$

$$\Rightarrow \hat{S}^2_{p_i} = S_y^2(1+e_0)\left(1-\delta_{p_i}e_1 + \delta_{p_i}^2 e_1^2 - \delta_{p_i}^3 e_1^3 + \cdots\right)$$

Expanding and neglecting the terms more than 3rd order, we get

$$\hat{S}^2_{p_i} = S_y^2 + S_y^2 e_0 - S_y^2\delta_{p_i}e_1 - S_y^2\delta_{p_i}e_0 e_1 + S_y^2\delta_{p_i}^2 e_1^2$$

$$\Rightarrow \hat{S}^2_{p_i} - S_y^2 = S_y^2 e_0 - S_y^2\delta_{p_i}e_1 - S_y^2\delta_{p_i}e_0 e_1 + S_y^2\delta_{p_i}^2 e_1^2 \qquad \text{(A)}$$

By taking expectation on both sides of (A), we get

$$E\left(\hat{S}^2_{p_i} - S_y^2\right) = S_y^2 E(e_0) - S_y^2\delta_{p_i}E(e_1) - S_y^2\delta_{p_i}E(e_0 e_1) + S_y^2\delta_{p_i}^2 E(e_1^2)$$

$$\text{Bias}\left(\hat{S}^2_{p_i}\right) = S_y^2\delta_{p_i}^2 E(e_1^2) - S_y^2\delta_{p_i}E(e_0 e_1)$$

$$\text{Bias}\left(\hat{S}^2_{p_i}\right) = \gamma S_y^2\,\delta_{p_i}\left(\delta_{p_i}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right) \qquad \text{(B)}$$

Squaring both sides of (A), neglecting the terms more than 2nd order and taking expectation, we get:

$$E\left(\hat{S}^2_{p_i} - S_y^2\right)^2 = S_y^4 E(e_0^2) + S_y^4\delta_{p_i}^2 E(e_1^2) - 2S_y^4\delta_{p_i}E(e_0 e_1)$$

$$\mathrm{MSE}\left(\hat{S}^2_{p_j}\right) = \gamma S_y^4\left(\left(\beta_{2(y)} - 1\right) + \delta_{p_i}^2\left(\beta_{2(X)} - 1\right) - 2\delta_{p_i}(\lambda_{22} - 1)\right) \qquad \text{(C)}$$

Appendix C

Comparison with that of SRSWOR sample variance

For $\mathbf{MSE}\left(\hat{\mathbf{S}}^2_{\mathbf{p_j}}\right) \le \mathbf{V}\left(\mathbf{s}_{\mathbf{y}}^2\right)$

$$\gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_{p_i}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{p_i}(\lambda_{22} - 1)\right] \le \gamma S_y^4\left(\beta_{2(y)} - 1\right)$$

$$\Rightarrow \left[\left(\beta_{2(y)} - 1\right) + \delta_{p_i}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{p_i}(\lambda_{22} - 1)\right] \le \left(\beta_{2(y)} - 1\right)$$

$$\Rightarrow \delta_{p_i}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{p_i}(\lambda_{22} - 1) \le 0$$

$$\Rightarrow \delta_{p_i}^2\left(\beta_{2(x)} - 1\right) \le 2\delta_{p_i}(\lambda_{22} - 1)$$

$$\Rightarrow \delta_{p_i}\left(\beta_{2(x)} - 1\right) \le 2(\lambda_{22} - 1)$$

$$\Rightarrow \delta_{p_i} \le 2\frac{(\lambda_{22} - 1)}{\left(\beta_{2(x)} - 1\right)}$$

$$\Rightarrow \frac{S_x^2}{S_x^2 + \alpha\omega_i} \le 2\frac{(\lambda_{22} - 1)}{\left(\beta_{2(x)} - 1\right)}$$

$$\Rightarrow \frac{\left(\beta_{2(x)} - 1\right)}{2(\lambda_{22} - 1)} \le \frac{S_x^2 + \alpha\omega_i}{S_x^2}$$

$$\Rightarrow S_x^2\left[\frac{\left(\beta_{2(x)} - 1\right)}{2(\lambda_{22} - 1)}\right] - S_x^2 \le \alpha\omega_i$$

$$\Rightarrow \omega_i^{-1}S_x^2\left[\frac{\left(\beta_{2(x)} - 1\right)}{2(\lambda_{22} - 1)} - 1\right] \le \alpha$$

$$\mathrm{MSE}\left(\hat{S}^2_{p_j}\right) \le \mathrm{V}\left(s_y^2\right) \text{ if } \alpha \ge \omega_i^{-1}S_x^2\left[\frac{\left(\beta_{2(x)} - 1\right)}{2(\lambda_{22} - 1)} - 1\right]$$

Appendix D

Comparison with that of the ratio type variance estimator

For $\mathbf{MSE}\left(\hat{\mathbf{S}}^2_{\mathbf{p_i}}\right) \le \mathbf{MSE}\left(\hat{\mathbf{S}}^2_{\mathbf{R}}\right)$

$$\gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_{p_i}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{p_i}(\lambda_{22} - 1)\right]$$
$$\le \gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \left(\beta_{2(x)} - 1\right) - 2(\lambda_{22} - 1)\right]$$

$$\Rightarrow \left(\beta_{2(y)}-1\right)+\delta_{p_i}^2\left(\beta_{2(x)}-1\right)-2\delta_{p_i}(\lambda_{22}-1) \leq \left(\beta_{2(y)}-1\right)+\left(\beta_{2(x)}-1\right)-2(\lambda_{22}-1)$$

$$\Rightarrow \delta_{p_i}^2\left(\beta_{2(x)}-1\right)-2\delta_{p_i}(\lambda_{22}-1) \leq \left(\beta_{2(x)}-1\right)-2(\lambda_{22}-1)$$

$$\Rightarrow \delta_{p_i}^2\left(\beta_{2(x)}-1\right)-2\delta_{p_i}(\lambda_{22}-1)-\left(\beta_{2(x)}-1\right)+2(\lambda_{22}-1) \leq 0$$

$$\Rightarrow \left(\delta_{p_i}^2-1\right)\left(\beta_{2(x)}-1\right)-2(\lambda_{22}-1)\left(\delta_{p_i}-1\right) \leq 0$$

$$\Rightarrow \left(\delta_{p_i}+1\right)\left(\delta_{p_i}-1\right)\left(\beta_{2(x)}-1\right)-2(\lambda_{22}-1)\left(\delta_{p_i}-1\right) \leq 0$$

$$\Rightarrow \left(\delta_{p_i}-1\right)\left[\left(\delta_{p_i}+1\right)\left(\beta_{2(x)}-1\right)-2(\lambda_{22}-1)\right] \leq 0$$

Condition 1: $\left(\delta_{p_i}-1\right) \leq 0$ and $\left(\delta_{p_i}+1\right)\left(\beta_{2(x)}-1\right)-2(\lambda_{22}-1) \geq 0$

$$\delta_{p_i}-1 \leq 0 \text{ and } \left(\delta_{p_i}+1\right)\left(\beta_{2(x)}-1\right) \geq 2(\lambda_{22}-1)$$

$$\Rightarrow \delta_{p_i} \leq 1 \text{ and } \left(\delta_{p_i}+1\right) \geq \frac{2(\lambda_{22}-1)}{\left(\beta_{2(x)}-1\right)}$$

$$\Rightarrow \delta_{p_i} \leq 1 \text{ and } \delta_{p_i} \geq \frac{2(\lambda_{22}-1)}{\left(\beta_{2(x)}-1\right)}-1$$

$$\Rightarrow \frac{2(\lambda_{22}-1)}{\left(\beta_{2(x)}-1\right)}-1 \leq \delta_{p_i} \leq 1$$

$$\Rightarrow \frac{2(\lambda_{22}-1)}{\left(\beta_{2(x)}-1\right)}-1 \leq \frac{S_x^2}{S_x^2+\alpha\omega_i} \leq 1$$

$$\Rightarrow \frac{2(\lambda_{22}-1)-\left(\beta_{2(x)}-1\right)}{\left(\beta_{2(x)}-1\right)} \leq \frac{S_x^2}{S_x^2+\alpha\omega_i} \leq 1$$

$$\Rightarrow \frac{2\left(\beta_{2(x)}-1\right)}{2(\lambda_{22}-1)-\left(\beta_{2(x)}-1\right)} \geq \frac{S_x^2+\alpha\omega_i}{S_x^2} \geq 1$$

$$\Rightarrow S_x^2\left[\frac{2\left(\beta_{2(x)}-1\right)}{2(\lambda_{22}-1)-\left(\beta_{2(x)}-1\right)}\right] \geq S_x^2+\alpha\omega_i \geq S_x^2$$

$$\Rightarrow S_x^2\left[\frac{2\left(\beta_{2(x)}-1\right)}{2(\lambda_{22}-1)-\left(\beta_{2(x)}-1\right)}\right]-S_x^2 \geq \alpha\omega_i \geq 0$$

$$\Rightarrow \omega_i^{-1}S_x^2\left[\frac{2\left(\beta_{2(x)}-1\right)}{2(\lambda_{22}-1)-\left(\beta_{2(x)}-1\right)}-1\right] \geq \alpha \geq 0$$

$$\Rightarrow 0 \le \alpha \le \omega_i^{-1} S_x^2 \left[\frac{2\left(\beta_{2(x)}-1\right)}{2(\lambda_{22}-1)-\left(\beta_{2(x)}-1\right)}-1\right]$$

$$\Rightarrow 0 \le \alpha \le \omega_i^{-1} S_x^2 \left[\frac{2\left(\beta_{2(x)}-1\right)-2(\lambda_{22}-1)}{2(\lambda_{22}-1)-\left(\beta_{2(x)}-1\right)}\right]$$

Condition 2: $\left(\delta_{p_i}-1\right) \ge 0$ and $\left(\delta_{p_i}+1\right)\left(\beta_{2(x)}-1\right)-2(\lambda_{22}-1) \le 0$

$$\delta_{p_i}-1 \ge 0 \text{ and } \left(\delta_{p_i}+1\right)\left(\beta_{2(x)}-1\right) \le 2(\lambda_{22}-1)$$

$$\Rightarrow \delta_{p_i} \ge 1 \text{ and } \left(\delta_{p_i}+1\right) \le \frac{2(\lambda_{22}-1)}{\left(\beta_{2(x)}-1\right)}$$

$$\Rightarrow \delta_{p_i} \ge 1 \text{ and } \delta_{p_i} \le \frac{2(\lambda_{22}-1)}{\left(\beta_{2(x)}-1\right)}-1$$

$$\Rightarrow \frac{2(\lambda_{22}-1)}{\left(\beta_{2(x)}-1\right)}-1 \ge \delta_{p_i} \ge 1$$

$$\Rightarrow \frac{2(\lambda_{22}-1)}{\left(\beta_{2(x)}-1\right)}-1 \ge \frac{S_x^2}{S_x^2+\alpha\omega_i} \ge 1$$

$$\Rightarrow \frac{2(\lambda_{22}-1)-\left(\beta_{2(x)}-1\right)}{\left(\beta_{2(x)}-1\right)} \ge \frac{S_x^2}{S_x^2+\alpha\omega_i} \ge 1$$

$$\Rightarrow \frac{2\left(\beta_{2(x)}-1\right)}{2(\lambda_{22}-1)-\left(\beta_{2(x)}-1\right)} \le \frac{S_x^2+\alpha\omega_i}{S_x^2} \le 1$$

$$\Rightarrow S_x^2 \left[\frac{2\left(\beta_{2(x)}-1\right)}{2(\lambda_{22}-1)-\left(\beta_{2(x)}-1\right)}\right] \le S_x^2+\alpha\omega_i \le S_x^2$$

$$\Rightarrow S_x^2 \left[\frac{2\left(\beta_{2(x)}-1\right)}{2(\lambda_{22}-1)-\left(\beta_{2(x)}-1\right)}\right]-S_x^2 \le \alpha\omega_i \le 0$$

$$\Rightarrow \omega_i^{-1} S_x^2 \left[\frac{2\left(\beta_{2(x)}-1\right)}{2(\lambda_{22}-1)-\left(\beta_{2(x)}-1\right)}-1\right] \le \alpha \le 0$$

$$\Rightarrow 0 \le \alpha \le \omega_i^{-1} S_x^2 \left[\frac{2\left(\beta_{2(x)}-1\right)}{2(\lambda_{22}-1)-\left(\beta_{2(x)}-1\right)}-1\right]$$

$$\Rightarrow \omega_i^{-1} S_x^2 \left[\frac{2\left(\beta_{2(x)}-1\right)-2(\lambda_{22}-1)}{2(\lambda_{22}-1)-\left(\beta_{2(x)}-1\right)}\right] \le \alpha \le 0$$

$$MSE(\hat{S}^2_{p_i}) < MSE(\hat{S}^2_R) \text{ if } 0 < \alpha < \omega_i^{-1} S_x^2 \left[\frac{2(\beta_{2(x)}-1)-2(\lambda_{22}-1)}{2(\lambda_{22}-1)-(\beta_{2(x)}-1)}\right]$$

$$\text{(or) } \omega_i^{-1} S_x^2 \left[\frac{2(\beta_{2(x)}-1)-2(\lambda_{22}-1)}{2(\lambda_{22}-1)-(\beta_{2(x)}-1)}\right] < \alpha < 0$$

Appendix E

Comparison with that of Existing modified ratio type variance Estimators
For $MSE(\hat{S}^2_{p_i}) \le MSE(\hat{S}^2_i)$

$$\gamma S_y^4 \left[\left(\beta_{2(y)}-1\right) + \delta_{p_i}^2 \left(\beta_{2(x)}-1\right) - 2\delta_{p_i}(\lambda_{22}-1)\right]$$
$$\le \gamma S_y^4 \left[\left(\beta_{2(y)}-1\right) + \delta_i^2 \left(\beta_{2(x)}-1\right) - 2\delta_i(\lambda_{22}-1)\right]$$

$$\Rightarrow \left(\beta_{2(y)}-1\right) + \delta_{p_i}^2 \left(\beta_{2(x)}-1\right) - 2\delta_{p_i}(\lambda_{22}-1)$$
$$\le \left(\beta_{2(y)}-1\right) + \delta_i^2 \left(\beta_{2(x)}-1\right) - 2\delta_i(\lambda_{22}-1)$$

$$\Rightarrow \delta_{p_i}^2 \left(\beta_{2(x)}-1\right) - 2\delta_{p_i}(\lambda_{22}-1) \le \delta_i^2 \left(\beta_{2(x)}-1\right) - 2\delta_i(\lambda_{22}-1)$$

$$\Rightarrow \delta_{p_i}^2 \left(\beta_{2(x)}-1\right) - 2\delta_{p_i}(\lambda_{22}-1) - \delta_i^2 \left(\beta_{2(x)}-1\right) + 2\delta_i(\lambda_{22}-1) \le 0$$

$$\Rightarrow \left(\delta_{p_i}^2 - \delta_i^2\right)\left(\beta_{2(x)}-1\right) - 2(\lambda_{22}-1)\left(\delta_{p_i}-\delta_i\right) \le 0$$

$$\Rightarrow \left(\delta_{p_i}+\delta_i\right)\left(\delta_{p_i}-\delta_i\right)\left(\beta_{2(x)}-1\right) - 2(\lambda_{22}-1)\left(\delta_{p_i}-\delta_i\right) \le 0$$

$$\Rightarrow \left(\delta_{p_i}-\delta_i\right)\left[\left(\delta_{p_i}+\delta_i\right)\left(\beta_{2(x)}-1\right) - 2(\lambda_{22}-1)\right] \le 0$$

Condition 1: $\left(\delta_{p_i}-\delta_i\right) \le 0$ and $\left(\delta_{p_i}+\delta_i\right)\left(\beta_{2(x)}-1\right) - 2(\lambda_{22}-1) \ge 0$

$$\delta_{p_i}-\delta_i \le 0 \text{ and } \left(\delta_{p_i}+\delta_i\right)\left(\beta_{2(x)}-1\right) \ge 2(\lambda_{22}-1)$$

$$\Rightarrow \delta_{p_i} \le \delta_i \text{ and } \left(\delta_{p_i}+\delta_i\right) \ge \frac{2(\lambda_{22}-1)}{\left(\beta_{2(x)}-1\right)}$$

$$\Rightarrow \delta_{p_i} \le \delta_i \text{ and } \delta_{p_i} \ge \frac{2(\lambda_{22}-1)}{\left(\beta_{2(x)}-1\right)} - \delta_i$$

$$\Rightarrow \frac{2(\lambda_{22}-1)}{\left(\beta_{2(x)}-1\right)} - \delta_i \le \delta_{p_i} \le \delta_i$$

$$\Rightarrow \frac{2(\lambda_{22}-1) - \delta_i\left(\beta_{2(x)}-1\right)}{\left(\beta_{2(x)}-1\right)} \le \frac{S_x^2}{S_x^2 + \alpha\omega_i} \le \delta_i$$

$$\Rightarrow \frac{\left(\beta_{2(x)}-1\right)}{2(\lambda_{22}-1) - \delta_i\left(\beta_{2(x)}-1\right)} \ge \frac{S_x^2 + \alpha\omega_i}{S_x^2} \ge \delta_i^{-1}$$

$$\Rightarrow \frac{\left(\beta_{2(x)}-1\right)}{2(\lambda_{22}-1)-\delta_i\left(\beta_{2(x)}-1\right)} \geq \frac{S_x^2+\alpha\omega_i}{S_x^2} \geq \delta_i^{-1}$$

$$\Rightarrow \omega_i^{-1}S_x^2\left[\frac{(1+\delta_i)\left(\beta_{2(x)}-1\right)-2(\lambda_{22}-1)}{2(\lambda_{22}-1)-\delta_i\left(\beta_{2(x)}-1\right)}\right] \geq \alpha \geq 1$$

Condition 2: $\left(\delta_{p_i}-\delta_i\right) \geq 0$ and $\left(\delta_{p_i}+\delta_i\right)\left(\beta_{2(x)}-1\right)-2(\lambda_{22}-1) \leq 0$

$$\delta_{p_i}-\delta_i \geq 0 \text{ and } \left(\delta_{p_i}+\delta_i\right)\left(\beta_{2(x)}-1\right) \leq 2(\lambda_{22}-1)$$

$$\Rightarrow \delta_{p_i} \geq \delta_i \text{ and } \left(\delta_{p_i}+\delta_i\right) \leq \frac{2(\lambda_{22}-1)}{\left(\beta_{2(x)}-1\right)}$$

$$\Rightarrow \delta_{p_i} \geq \delta_i \text{ and } \delta_{p_i} \leq \frac{2(\lambda_{22}-1)}{\left(\beta_{2(x)}-1\right)}-\delta_i$$

$$\Rightarrow \delta_i \leq \delta_{p_i} \leq \frac{2(\lambda_{22}-1)}{\left(\beta_{2(x)}-1\right)}-\delta_i$$

$$\Rightarrow \frac{2(\lambda_{22}-1)-\delta_i\left(\beta_{2(x)}-1\right)}{\left(\beta_{2(x)}-1\right)} \geq \frac{S_x^2}{S_x^2+\alpha\omega_i} \geq \delta_i$$

$$\Rightarrow \frac{\left(\beta_{2(x)}-1\right)}{2(\lambda_{22}-1)-\delta_i\left(\beta_{2(x)}-1\right)} \leq \frac{S_x^2+\alpha\omega_i}{S_x^2} \leq \delta_i^{-1}$$

$$\Rightarrow \frac{\left(\beta_{2(x)}-1\right)}{2(\lambda_{22}-1)-\delta_i\left(\beta_{2(x)}-1\right)} \leq \frac{S_x^2+\alpha\omega_i}{S_x^2} \leq \delta_i^{-1}$$

$$\Rightarrow \omega_i^{-1}S_x^2\left[\frac{(1+\delta_i)\left(\beta_{2(x)}-1\right)-2(\lambda_{22}-1)}{2(\lambda_{22}-1)-\delta_i\left(\beta_{2(x)}-1\right)}\right] \leq \alpha \leq 1$$

$$\text{MSE}\left(\hat{S}_{p_i}^2\right) < \text{MSE}\left(\hat{S}_i^2\right) \text{ if } 1 < \alpha < \omega_i^{-1}S_x^2\left[\frac{(1+\delta_i)\left(\beta_{2(x)}-1\right)-2(\lambda_{22}-1)}{2(\lambda_{22}-1)-\delta_i\left(\beta_{2(x)}-1\right)}\right]$$

$$\text{(or) } \omega_i^{-1}S_x^2\left[\frac{(1+\delta_i)\left(\beta_{2(x)}-1\right)-2(\lambda_{22}-1)}{2(\lambda_{22}-1)-\delta_i\left(\beta_{2(x)}-1\right)}\right] < \alpha < 1$$