A Generalized Class of Estimators of The Ratio of Two Population Means Using Auxiliary Information

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ABSTRACT

Using known population mean of an auxiliary variable, a generalized class of estimators of population ratio of two population means of study variables is proposed, its bias and mean square error are found and comparative aspects have also been studied and given.

Keywords: Generalized class of estimators, auxiliary variable, bias, mean square error, estimated optimum values and efficiency.

1. Introduction

Statisticians often make use of the information available on an auxiliary variable which is highly correlated with the variable under study for improving the efficiency of an estimator. Some examples of estimation of ratio estimators are the estimation yield rates in a crop survey, input-output ratios in an industrial survey, the ratio of corn acres to wheat acres, the ratio of expenditures on labour to total expenditures, or the ratio of liquid assets to total assets.

In order to have a better understanding of the ratio method of estimation (Cochran, 1977; Murthy, 1967), let us consider an example where y may denote the number of bullocks on a holding and x its area in acres, the ratio R_n giving an estimate of the number of bullocks per acre of a holding in the population. The product of R_n with the mean acreage of the holdings gives an estimate of the mean number of bullocks per holding.

Let (y_1, y_2) be the two study variables of interest with their population means $(\bar{Y}_1, \bar{Y}_2)$ respectively, and x be the auxiliary variable with known population mean $\bar{X}$. Let (Y_{1i}, Y_{2i}, X_i) be the ith (i=1,2,…,N) values on the variables

(y_1, y_2, x) in the population of size N respectively, $(\rho_{10}, \rho_{20}, \rho_{12})$ be the correlation coefficients between $\{(y_1, x), (y_2, x), (y_1, y_2)\}$ and (C_1^2, C_2^2, C_0^2) be the coefficients of variation of (y_1, y_2, x) respectively.

Further, let

$$\overline{Y}_1 = \frac{1}{N}\sum_{i=1}^{N} Y_{1i} , \qquad \overline{Y}_2 = \frac{1}{N}\sum_{i=1}^{N} Y_{2i} , \qquad \overline{X} = \frac{1}{N}\sum_{i=1}^{N} X_i ,$$

$$S_{y_1}^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(Y_{1i} - \overline{Y}_1)^2 , \qquad S_{y_2}^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(Y_{2i} - \overline{Y}_2)^2 ,$$

$$S_x^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(X_i - \overline{X})^2 , \qquad C_1^2 = \frac{S_{y_1}^2}{\overline{Y}_1^2} , \qquad C_2^2 = \frac{S_{y_2}^2}{\overline{Y}_2^2} ,$$

$$C_0^2 = \frac{S_x^2}{\overline{X}^2} , \quad \rho_{20} = \frac{S_{y_2x}}{S_{y_2}S_x} , \quad S_{y_1x} = \frac{1}{(N-1)}\sum_{i=1}^{N}(Y_{1i} - \overline{Y}_1)(X_i - \overline{X}) ,$$

$$S_{y_2x} = \frac{1}{(N-1)}\sum_{i=1}^{N}(Y_{2i} - \overline{Y}_2)(X_i - \overline{X}) , \qquad \rho_{10} = \frac{S_{y_1x}}{S_{y_1}S_x} ,$$

$$S_{y_1y_2} = \frac{1}{(N-1)}\sum_{i=1}^{N}(Y_{1i} - \overline{Y}_1)(Y_{2i} - \overline{Y}_2) , \qquad \rho_{12} = \frac{S_{y_1y_2}}{S_{y_1}S_{y_2}} .$$

Also, a simple random sample of size n without replacement is taken from the population of size N and let (y_{1i}, y_{2i}, x_i) be the observations on (y_1, y_2, x) respectively for the i[th] (i=1,2,…,n) selected unit in the sample of size n. Now, let

$$\overline{y}_1 = \frac{1}{n}\sum_{i=1}^{n} y_{1i} , \qquad \overline{y}_2 = \frac{1}{n}\sum_{i=1}^{n} y_{2i} , \qquad \overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i ,$$

$$s_{y_1}^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}(y_{1i} - \overline{y}_1)^2 , \qquad s_{y_2}^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}(y_{2i} - \overline{y}_2)^2 ,$$

$$s_x^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}(x_i - \overline{x})^2 , \qquad s_{y_1x} = \frac{1}{(n-1)}\sum_{i=1}^{n}(y_{1i} - \overline{y}_1)(x_i - \overline{x}) ,$$

$$s_{y_2x} = \frac{1}{(n-1)}\sum_{i=1}^{n}(y_{2i} - \overline{y}_2)(x_i - \overline{x}) ,$$

$$s_{y_1y_2} = \frac{1}{(n-1)}\sum_{i=1}^{n}(y_{1i} - \overline{y}_1)(y_{2i} - \overline{y}_2) .$$

Some particular estimators of ratio $R=\frac{\bar{Y}_1}{\bar{Y}_2}$ of two population means $\bar{Y}_1$ and $\bar{Y}_2$ using known mean $\bar{X}$ of the auxiliary variable x are given as follows:

$$\hat{R}_1=\frac{\bar{y}_1}{\bar{y}_2}+k(\bar{x}-\bar{X})=\hat{R}+k(\bar{x}-\bar{X}) \tag{1}$$

$$\hat{R}_2=\frac{\bar{y}_1+k(\bar{x}-\bar{X})}{\bar{y}_2} \tag{2}$$

$$\hat{R}_3=\frac{\bar{y}_1}{\bar{y}_2}.\frac{\bar{x}}{\bar{X}} \tag{3}$$

$$\hat{R}_4=\frac{\bar{y}_1}{\bar{y}_2}.\frac{\bar{X}}{\bar{x}} \tag{4}$$

$$\hat{R}_5=\frac{\bar{y}_1+k(\bar{x}-\bar{X})}{\bar{y}_2+(\bar{x}-\bar{X})} \tag{5}$$

$$\hat{R}_6=\frac{\bar{y}_1+k(\bar{x}-\bar{X})}{\bar{y}_2}\left(\frac{\bar{x}}{\bar{X}}\right) \tag{6}$$

where k is the characterizing scalar to be chosen suitably.

Seeing the forms of the estimators given in (1) to (6), we define a generalized class of estimators of the population ratio $R=\frac{\bar{Y}_1}{\bar{Y}_2}$ to be the bounded function $\hat{R}_g=g\left(\bar{y}_1,\bar{y}_2,\bar{x}\right)$ satisfying the Taylor's series expansion and $\hat{R}_g=g\left(\bar{y}_1,\bar{y}_2,\bar{x}\right)$ at the point $P=\left(\bar{Y}_1,\bar{Y}_2,\bar{X}\right)$ is such that

(i) The value of the function $g\left(\bar{y}_1,\bar{y}_2,\bar{x}\right)$ at point $\left(\bar{Y}_1,\bar{Y}_2,\bar{X}\right)$ is

$$g\left(\bar{Y}_1,\bar{Y}_2,\bar{X}\right)=R \tag{7}$$

(ii) The first partial derivatives of $g\left(\bar{y}_1,\bar{y}_2,\bar{x}\right)$ with respect to $\bar{y}_1$ and $\bar{y}_2$

$$g_1 = \frac{\partial g\left(\bar{y}_1, \bar{y}_2, \bar{x}\right)}{\partial \bar{y}_1}\Bigg]_{P=\left(\bar{Y}_1, \bar{Y}_2, \bar{X}\right)} = \frac{1}{\bar{Y}_2} \qquad (8)$$

and

$$g_2 = \frac{\partial g\left(\bar{y}_1, \bar{y}_2, \bar{x}\right)}{\partial \bar{y}_2}\Bigg]_{P=\left(\bar{Y}_1, \bar{Y}_2, \bar{X}\right)} = -\frac{\bar{Y}_1}{\bar{Y}_2^2} \qquad (9)$$

(iii) The second order partial derivatives of $g\left(\bar{y}_1, \bar{y}_2, \bar{x}\right)$ with respect to $\bar{y}_1$ and $\bar{y}_2$

$$g_{11} = \frac{\partial^2 g\left(\bar{y}_1, \bar{y}_2, \bar{x}\right)}{\partial \bar{y}_1^2}\Bigg]_{P=\left(\bar{Y}_1, \bar{Y}_2, \bar{X}\right)} = 0 \qquad (10)$$

and

$$g_{22} = \frac{\partial^2 g\left(\bar{y}_1, \bar{y}_2, \bar{x}\right)}{\partial \bar{y}_2^2}\Bigg]_{P=\left(\bar{Y}_1, \bar{Y}_2, \bar{X}\right)} = \frac{2\bar{Y}_1}{\bar{Y}_2^3} \qquad (11)$$

(iv) The second order partial derivatives of $g\left(\bar{y}_1, \bar{y}_2, \bar{x}\right)$ with respect to $\left(\bar{y}_1, \bar{y}_2\right)$ and $\left(\bar{y}_2, \bar{y}_1\right)$

$$g_{12} = \frac{\partial^2 g\left(\bar{y}_1, \bar{y}_2, \bar{x}\right)}{\partial \bar{y}_1 \partial \bar{y}_2}\Bigg]_{P=\left(\bar{Y}_1, \bar{Y}_2, \bar{X}\right)} = -\frac{1}{\bar{Y}_2^2} \qquad (12)$$

and

$$g_{21} = \frac{\partial^2 g\left(\bar{y}_1, \bar{y}_2, \bar{x}\right)}{\partial \bar{y}_2 \partial \bar{y}_1}\Bigg]_{P=\left(\bar{Y}_1, \bar{Y}_2, \bar{X}\right)} = -\frac{1}{\bar{Y}_2^2} \qquad (13)$$

(v) The first and the second order partial derivatives of $g\left(\bar{y}_1, \bar{y}_2, \bar{x}\right)$ with respect to $\bar{x}$ are

$$g_0 = \left.\frac{\partial g\left(\bar{y}_1, \bar{y}_2, \bar{x}\right)}{\partial \bar{x}}\right]_{P=\left(\bar{Y}_1, \bar{Y}_2, \bar{X}\right)} \tag{14}$$

and

$$g_{00} = \left.\frac{\partial^2 g\left(\bar{y}_1, \bar{y}_2, \bar{x}\right)}{\partial \bar{x}^2}\right]_{P=\left(\bar{Y}_1, \bar{Y}_2, \bar{X}\right)} \tag{15}$$

It may be mentioned here that all estimators from (1) to (6) satisfy all the conditions of $\hat{R}_g = g\left(\bar{y}_1, \bar{y}_2, \bar{x}\right)$ and hence, they all belong to general class of estimators $\hat{R}_g$ and the results of (1) to (6) may be easily found as the special cases of the general result $\hat{R}_g$.

2. Bias and Mean Square Error

Let us define,

$$e_0 = \frac{\left(\bar{x} - \bar{X}\right)}{\bar{X}} \ , \ e_1 = \frac{\left(\bar{y}_1 - \bar{Y}_1\right)}{\bar{Y}_1} \ , \ e_2 = \frac{\left(\bar{y}_2 - \bar{Y}_2\right)}{\bar{Y}_2}$$

so that $E\left(e_0\right) = E\left(e_1\right) = E\left(e_2\right) = 0$ and

$$E\left(e_0^2\right) = \frac{\left(N-n\right)}{Nn} C_0^2 \ ,$$

$$E\left(e_1^2\right) = \frac{\left(N-n\right)}{Nn} C_1^2 \ ,$$

$$E\left(e_2^2\right) = \frac{\left(N-n\right)}{Nn} C_2^2 \ ,$$

$$E\left(e_0 e_1\right) = \frac{\left(N-n\right)}{Nn} \rho_{10} C_1 C_0 \ ,$$

$$E\left(e_0 e_2\right) = \frac{\left(N-n\right)}{Nn} \rho_{20} C_2 C_0 \ ,$$

$$E\left(e_1 e_2\right) = \frac{\left(N-n\right)}{Nn} \rho_{12} C_1 C_2 \ .$$

Expanding $\hat{R}_g = g\left(\bar{y}_1, \bar{y}_2, \bar{x}\right)$ about the point $P = \left(\bar{Y}_1, \bar{Y}_2, \bar{X}\right)$ in third order Taylor's series, we have

$$\hat{R}_g = g\left(\bar{Y}_1,\bar{Y}_2,\bar{X}\right)+\left(\bar{y}_1-\bar{Y}_1\right)g_1+\left(\bar{y}_2-\bar{Y}_2\right)g_2+\left(\bar{x}-\bar{X}\right)g_0$$
$$+\frac{1}{2!}\Big\{\left(\bar{y}_1-\bar{Y}_1\right)^2 g_{11}+\left(\bar{y}_2-\bar{Y}_2\right)^2 g_{22}+\left(\bar{x}-\bar{X}\right)^2 g_{00}+2\left(\bar{y}_1-\bar{Y}_1\right)\left(\bar{y}_2-\bar{Y}_2\right)g_{12}$$
$$+2\left(\bar{y}_1-\bar{Y}_1\right)\left(\bar{x}-\bar{X}\right)g_{10}+2\left(\bar{y}_2-\bar{Y}_2\right)\left(\bar{x}-\bar{X}\right)g_{20}\Big\}+\frac{1}{3!}^{2}\Big\{\left(\bar{y}_1-\bar{Y}_1\right)\frac{\partial}{\partial\bar{y}_1}$$
$$+\left(\bar{y}_2-\bar{Y}_2\right)\frac{\partial}{\partial\bar{y}_2}++\left(\bar{x}-\bar{X}\right)\frac{\partial}{\partial\bar{x}}\Big\}^3 g\left(\bar{y}_{1*},\bar{y}_{2*},\bar{x}_*\right) \tag{16}$$

where $\bar{y}_{1*}=\bar{Y}_1+\theta\left(\bar{y}_1-\bar{Y}_1\right)$, $\bar{y}_{2*}=\bar{Y}_2+\theta\left(\bar{y}_2-\bar{Y}_2\right)$ and $\bar{x}_*=\bar{X}+\theta\left(\bar{x}-\bar{X}\right)$ for $0<\theta<1$.

On substituting $g\left(\bar{Y}_1,\bar{Y}_2,\bar{X}\right)=R$ and the values of the derivatives in (16), we get

$$\hat{R}_g = R+\left(\bar{y}_1-\bar{Y}_1\right)\left(\frac{1}{\bar{Y}_2}\right)+\left(\bar{y}_2-\bar{Y}_2\right)\left(-\frac{\bar{Y}_1}{\bar{Y}_2^2}\right)+\left(\bar{x}-\bar{X}\right)g_0+\frac{1}{2!}\Big\{\left(\bar{y}_1-\bar{Y}_1\right)^2.0$$
$$+\left(\bar{y}_2-\bar{Y}_2\right)^2\left(2\frac{\bar{Y}_1}{\bar{Y}_2^3}\right)+\left(\bar{x}-\bar{X}\right)^2 g_{00}+2\left(\bar{y}_1-\bar{Y}_1\right)\left(\bar{y}_2-\bar{Y}_2\right)\left(-\frac{1}{\bar{Y}_2^2}\right)$$
$$+2\left(\bar{y}_1-\bar{Y}_1\right)\left(\bar{x}-\bar{X}\right)g_{10}+2\left(\bar{y}_2-\bar{Y}_2\right)\left(\bar{x}-\bar{X}\right)g_{20}\Big\}+\frac{1}{3!}\Big\{\left(\bar{y}_1-\bar{Y}_1\right)\frac{\partial}{\partial\bar{y}_1}$$
$$+\left(\bar{y}_2-\bar{Y}_2\right)\frac{\partial}{\partial\bar{y}_2}+\left(\bar{x}-\bar{X}\right)\frac{\partial}{\partial\bar{x}}\Big\}^3 g\left(\bar{y}_{1*},\bar{y}_{2*},\bar{x}_*\right) \tag{17}$$

$$= R+\mathrm{Re}_1-\mathrm{Re}_2+\bar{X}e_0g_0+\frac{1}{2!}\Big\{2Re_2^2+\bar{X}^2e_0^2g_{00}-2R\,\mathrm{e}_1\mathrm{e}_2+2\bar{Y}_1\bar{X}\;\mathrm{e}_1\mathrm{e}_0\;g_{10}$$
$$+2\bar{Y}_2\bar{X}e_2e_0g_{20}\Big\}+\frac{1}{3!}\Big\{\bar{Y}_1e_1\frac{\partial}{\partial\bar{y}_1}+\bar{Y}_2e_2\frac{\partial}{\partial\bar{y}_2}+\bar{X}e_0\frac{\partial}{\partial\bar{x}}\Big\}^3 g\left(\bar{y}_{1*},\bar{y}_{2*},\bar{x}_*\right) \tag{18}$$

Taking expectation on both sides of (18) and ignoring terms in powers of e_i's (i=0,1,2) greater than two, we have bias to the first degree of approximation {that is, up to terms of order $O\left(\frac{1}{n}\right)$} to be

$$E\left(\hat{R}_g\right)-R=RE\left(e_1\right)-RE\left(e_2\right)+\bar{X}E\left(e_0\right)g_0+\frac{1}{2!}\Big\{2RE\left(e_2^2\right)+\bar{X}^2E\left(e_0^2\right)g_{00}$$
$$-2RE\left(e_1e_2\right)+2\bar{Y}_1\bar{X}E\left(e_0e_1\right)g_{10}+2\bar{Y}_2\bar{X}E\left(e_0e_2\right)g_{20}\Big\} \quad (19)$$

Or $$Bias(\hat{R}_g)=\left(\frac{1}{n}-\frac{1}{N}\right)\Big[RC_2^2+\frac{\bar{X}^2C_0^2}{2}g_{00}-R\rho_{12}C_1C_2+\bar{Y}_1\bar{X}\rho_{10}C_1C_0g_{10}$$
$$+\bar{Y}_2\bar{X}\rho_{20}C_2C_0g_{20}\Big] \quad (20)$$

which shows that bias of $\hat{R}_g$ is of order $O\left(\frac{1}{n}\right)$; hence, for sufficiently large value of n, the bias is negligible.

Squaring both the sides of (18), taking expectation and ignoring terms in powers of e_i's greater than two, the mean square error of $\hat{R}_g$ to the first degree of approximation {that is, up to terms of order $O\left(\frac{1}{n}\right)$} is

$$E(\hat{R}_g-R)^2=E\left[R(e_1-e_2)+\bar{X}e_0g_0\right]^2$$
$$=E\left[\left\{R(e_1-e_2)\right\}^2+\bar{X}^2e_0^2g_0^2+2R(e_1-e_2)\bar{X}e_0g_0\right]$$
$$=R^2E(e_1-e_2)^2+\bar{X}^2E\left(e_0^2\right)g_0^2+2R\bar{X}E\left(e_0e_1-e_0e_2\right)g_0$$
$$=R^2\left\{E\left(e_1^2\right)+E\left(e_2^2\right)-2E\left(e_1e_2\right)\right\}+\bar{X}^2E\left(e_0^2\right)g_0^2$$
$$+2R\bar{X}\left\{E\left(e_0e_1\right)-E\left(e_0e_2\right)\right\}g_0$$
$$=\left(\frac{1}{n}-\frac{1}{N}\right)R^2\left[C_1^2+C_2^2-2\rho_{12}C_1C_2\right]+\left(\frac{1}{n}-\frac{1}{N}\right)\bar{X}^2C_0^2g_0^2$$
$$+2R\bar{X}\left(\frac{1}{n}-\frac{1}{N}\right)\left(\rho_{10}C_1C_0-\rho_{20}C_2C_0\right)g_0$$
$$=\left(\frac{1}{n}-\frac{1}{N}\right)R^2\left[C_1^2+C_2^2-2\rho_{12}C_1C_2\right]$$
$$+\left(\frac{1}{n}-\frac{1}{N}\right)\bar{X}^2C_0^2\left\{g_0^2+\frac{2R}{\bar{X}}\left(\rho_{10}\frac{C_1}{C_0}-\rho_{20}\frac{C_2}{C_0}\right)g_0\right\}$$

$$MSE\left(\hat{R}_g\right)=MSE(\hat{R})+\left(\frac{1}{n}-\frac{1}{N}\right)C_0^2\bar{X}^2\left(g_0^2+\frac{2Rg_0}{\bar{X}}D\right) \quad (21)$$

where $\hat{R} = \frac{\bar{y}_1}{\bar{y}_2}$ is the ratio estimator of $R = \frac{\bar{Y}_1}{\bar{Y}_2}$ and $D = \rho_{10}\left(\frac{C_1}{C_0}\right) - \rho_{20}\left(\frac{C_2}{C_0}\right)$

In (21), the optimum value of g_0 minimizing the mean square error of $\hat{R}_g$ is given by

$$g_0 = -\left(\frac{RD}{\bar{X}}\right) = G \tag{22}$$

and the minimum mean square error of $\hat{R}_g$ is

$$MSE(\hat{R}_g)_{\min} = MSE\left(\hat{R}\right) + \left(\frac{1}{n} - \frac{1}{N}\right) C_0^2 \bar{X}^2 \left(\frac{R^2 D^2}{\bar{X}^2} - \frac{2R^2 D^2}{\bar{X}^2}\right)$$

$$= MSE(\hat{R}) - \left(\frac{1}{n} - \frac{1}{N}\right) C_0^2 R^2 D^2 \tag{23}$$

From (22) and (23), it is clear that $MSE(\hat{R}_g)_{\min}$ is attained if $\hat{R}_g$ satisfies the (i) to (v) conditions of $\hat{R}_g$ along with the condition $g_0 = -\left(\frac{RD}{\bar{X}}\right) = G$ in (22) which means $\hat{R}_g$ should not be function of $\left(\bar{y}_1, \bar{y}_2, \bar{x}\right)$ only but $g_0 = -\left(\frac{RD}{\bar{X}}\right) = G$ also to attain the minimum mean square error in (23). This means, we should have the estimator $g\left(\bar{y}_1, \bar{y}_2, \bar{x}, G\right)$ satisfying the conditions (i) to (v) and (22) to get the minimum mean square error in (23), but $g_0 = -\left(\frac{RD}{\bar{X}}\right) = G$ is unknown; hence, it should be estimated by $\hat{G}$ based on sample values and we get the estimator based on estimated optimum value $\hat{G}$ and thus we have the estimator

$$\hat{R}_{ge} = g\left(\bar{y}_1, \bar{y}_2, \bar{x}, \hat{G}\right) \tag{24}$$

where $\hat{G} = -\frac{\hat{R}\hat{D}}{\bar{x}}$

$$= -\frac{\hat{R}}{\bar{x}}\left(\hat{\rho}_{10}\frac{s_{y_1}}{\bar{y}_1}\frac{\bar{x}}{s_x} - \hat{\rho}_{20}\frac{s_{y_2}}{\bar{y}_2}\frac{\bar{x}}{s_x}\right)$$

$$= -\frac{\bar{y}_1}{\bar{y}_2 s_x^2}\left(\frac{s_{y_1 x}}{\bar{y}_1} - \frac{s_{y_2 x}}{\bar{y}_2}\right) \tag{25}$$

with $s_x^2 = \frac{1}{(n-1)} \sum_{i=1}^{n} (x_i - \bar{x})^2$, $s_{y_1 x} = \frac{1}{(n-1)} \sum_{i=1}^{n} (y_{1i} - \bar{y}_1)(x_i - \bar{x})$

and $s_{y_2 x} = \frac{1}{(n-1)} \sum_{i=1}^{n} (y_{2i} - \bar{y}_2)(x_i - \bar{x})$.

Now we find the conditions on $\hat{R}_{ge} = g(\bar{y}_1, \bar{y}_2, \bar{x}, \hat{G})$ such that $\hat{R}_{ge}$ attains the minimum mean square error of $\hat{R}_{ge}$ in (23).

Expanding $\hat{R}_{ge} = g(\bar{y}_1, \bar{y}_2, \bar{x}, \hat{G})$ about the point $P_e = (\bar{Y}_1, \bar{Y}_2, \bar{X}, G)$ in Taylor's series, we have

$$\begin{aligned}\hat{R}_{ge} &= g(\bar{Y}_1, \bar{Y}_2, \bar{X}, G) + (\bar{y}_1 - \bar{Y}_1) g_{1e} + (\bar{y}_2 - \bar{Y}_2) g_{2e} + (\bar{x} - \bar{X}) g_{0e} \\ &+ (\hat{G} - G) g_{3e} + \frac{1}{2!}\Big\{ (\bar{y}_1 - \bar{Y}_1)^2 g_{11e} + (\bar{y}_2 - \bar{Y}_2)^2 g_{22e} + (\bar{x} - \bar{X})^2 g_{00e} \\ &+ (\hat{G} - G)^2 g_{33e} + 2(\bar{y}_1 - \bar{Y}_1)(\bar{y}_2 - \bar{Y}_2) g_{12e} + 2(\bar{y}_1 - \bar{Y}_1)(\bar{x} - \bar{X}) g_{10e} \\ &+ 2(\bar{y}_1 - \bar{Y}_1)(\hat{G} - G) g_{13e} + 2(\bar{y}_2 - \bar{Y}_2)(\bar{x} - \bar{X}) g_{20e} \\ &+ 2(\bar{y}_2 - \bar{Y}_2)(\hat{G} - G) g_{23e} + 2(\bar{x} - \bar{X})(\hat{G} - G) g_{03e} \Big\} + \ldots \end{aligned} \quad (26)$$

Similar to $\hat{R}_g$, from (i) to (iv) and (22), substituting in (26),

$$g(\bar{Y}_1, \bar{Y}_2, \bar{X}, G) = R \quad (27)$$

$$g_{1e} = \left. \frac{\partial g(\bar{y}_1, \bar{y}_2, \bar{x}, \hat{G})}{\partial \bar{y}_1} \right]_{(\bar{Y}_1, \bar{Y}_2, \bar{X}, G)} = \frac{1}{\bar{Y}_2} \quad (28)$$

$$g_{2e} = \left. \frac{\partial g(\bar{y}_1, \bar{y}_2, \bar{x}, \hat{G})}{\partial \bar{y}_2} \right]_{(\bar{Y}_1, \bar{Y}_2, \bar{X}, G)} = -\frac{\bar{Y}_1}{\bar{Y}_2^2} \quad (29)$$

$$g_{0e} = \left. \frac{\partial g(\bar{y}_1, \bar{y}_2, \bar{x}, \hat{G})}{\partial \bar{x}} \right]_{(\bar{Y}_1, \bar{Y}_2, \bar{X}, G)} \quad (30)$$

$$g_{3e} = \left.\frac{\partial g\left(\bar{y}_1, \bar{y}_2, \bar{x}, \hat{G}\right)}{\partial \hat{G}}\right]_{\left(\bar{Y}_1, \bar{Y}_2, \bar{X}, G\right)} \tag{31}$$

$$g_{11e} = \left.\frac{\partial^2 g\left(\bar{y}_1, \bar{y}_2, \bar{x}, \hat{G}\right)}{\partial \bar{y}_1^2}\right]_{\left(\bar{Y}_1, \bar{Y}_2, \bar{X}, G\right)} = 0 \tag{32}$$

$$g_{22e} = \left.\frac{\partial^2 g\left(\bar{y}_1, \bar{y}_2, \bar{x}, \hat{G}\right)}{\partial \bar{y}_2^2}\right]_{\left(\bar{Y}_1, \bar{Y}_2, \bar{X}, G\right)} = \frac{2\bar{Y}_1}{\bar{Y}_2^3} \tag{33}$$

$$g_{00e} = \left.\frac{\partial^2 g\left(\bar{y}_1, \bar{y}_2, \bar{x}, \hat{G}\right)}{\partial \bar{x}^2}\right]_{\left(\bar{Y}_1, \bar{Y}_2, \bar{X}, G\right)} \tag{34}$$

$$g_{33e} = \left.\frac{\partial^2 g\left(\bar{y}_1, \bar{y}_2, \bar{x}, \hat{G}\right)}{\partial \hat{G}^2}\right]_{\left(\bar{Y}_1, \bar{Y}_2, \bar{X}, G\right)} \tag{35}$$

$$g_{12e} = \left.\frac{\partial^2 g\left(\bar{y}_1, \bar{y}_2, \bar{x}, \hat{G}\right)}{\partial \bar{y}_1 \partial \bar{y}_2}\right]_{\left(\bar{Y}_1, \bar{Y}_2, \bar{X}, G\right)} = -\frac{1}{\bar{Y}_2^2} \tag{36}$$

$$g_{10e} = \left.\frac{\partial^2 g\left(\bar{y}_1, \bar{y}_2, \bar{x}, \hat{G}\right)}{\partial \bar{y}_1 \partial \bar{x}}\right]_{\left(\bar{Y}_1, \bar{Y}_2, \bar{X}, G\right)} \tag{37}$$

$$g_{20e} = \left.\frac{\partial^2 g\left(\bar{y}_1, \bar{y}_2, \bar{x}, \hat{G}\right)}{\partial \bar{y}_2 \partial \bar{x}}\right]_{\left(\bar{Y}_1, \bar{Y}_2, \bar{X}, G\right)} \tag{38}$$

$$g_{13e} = \left.\frac{\partial^2 g\left(\bar{y}_1, \bar{y}_2, \bar{x}, \hat{G}\right)}{\partial \bar{y}_1 \partial \hat{G}}\right]_{\left(\bar{Y}_1, \bar{Y}_2, \bar{X}, G\right)} \tag{39}$$

$$g_{23e} = \left.\frac{\partial^2 g\left(\bar{y}_1, \bar{y}_2, \bar{x}, \hat{G}\right)}{\partial \bar{y}_2 \partial \hat{G}}\right]_{\left(\bar{Y}_1, \bar{Y}_2, \bar{X}, G\right)} \tag{40}$$

$$g_{03e} = \left.\frac{\partial^2 g\left(\bar{y}_1, \bar{y}_2, \bar{x}, \hat{G}\right)}{\partial \bar{x} \partial \hat{G}}\right]_{\left(\bar{Y}_1, \bar{Y}_2, \bar{X}, G\right)} \quad (41)$$

We have

$$\hat{R}_{ge} - R = \left(\bar{y}_1 - \bar{Y}_1\right)\frac{1}{\bar{Y}_2} - \left(\bar{y}_2 - \bar{Y}_2\right)\frac{\bar{Y}_1}{\bar{Y}_2^2} - \left(\bar{x} - \bar{X}\right)\frac{RD}{\bar{X}}$$

$$+\left(\hat{G}-G\right)g_{3e} + \frac{1}{2!}\Big\{\left(\bar{y}_1 - \bar{Y}_1\right)^2.0 + \left(\bar{y}_2 - \bar{Y}_2\right)^2\frac{2\bar{Y}_1}{\bar{Y}_2^3} + \left(\bar{x} - \bar{X}\right)^2 g_{00e}$$

$$+\left(\hat{G}-G\right)^2 g_{33e} - 2\left(\bar{y}_1 - \bar{Y}_1\right)\left(\bar{y}_2 - \bar{Y}_2\right)\frac{1}{\bar{Y}_2^2} + 2\left(\bar{y}_1 - \bar{Y}_1\right)\left(\bar{x} - \bar{X}\right)g_{10e}$$

$$+2\left(\bar{y}_1 - \bar{Y}_1\right)\left(\hat{G}-G\right)g_{13e} + 2\left(\bar{y}_2 - \bar{Y}_2\right)\left(\bar{x} - \bar{X}\right)g_{20e}$$

$$+2\left(\bar{y}_2 - \bar{Y}_2\right)\left(\hat{G}-G\right)g_{23e} + 2\left(\bar{x} - \bar{X}\right)\left(\hat{G}-G\right)g_{03e}\Big\} + \ldots$$

$$= \mathrm{Re}_1 - \mathrm{Re}_2 - RDe_0 + \left(\hat{G}-G\right)g_{3e} + \frac{1}{2!}\Big\{2Re_2^2 + \bar{X}^2 e_0^2 g_{00e}$$

$$+\left(\hat{G}-G\right)^2 g_{33e} - 2R\,\mathrm{e}_1\mathrm{e}_2 + 2\bar{Y}_1\bar{X}\,\mathrm{e}_1\mathrm{e}_0\, g_{10e} + 2\bar{Y}_1\,\mathrm{e}_1\left(\hat{G}-G\right)g_{13e}$$

$$+2\bar{Y}_2\bar{X}e_2e_0g_{20e} + 2\bar{Y}_2\,\mathrm{e}_2\left(\hat{G}-G\right)g_{23e} + 2\bar{X}e_0\left(\hat{G}-G\right)g_{03e}\Big\} + \ldots \quad (42)$$

The mean square error $E\left(\hat{R}_{ge} - R\right)^2$ to the first degree of approximation is

$$E(\hat{R}_{ge} - R)^2 = E\left[R(e_1 - e_2 - e_0 D) + \left(\hat{G}-G\right)g_{3e}\right]^2$$

$$= E\Big[\left\{R(e_1 - e_2) - \mathrm{Re}_0\, D\right\}^2 + \left(\hat{G}-G\right)^2 g_{3e}^2$$

$$+\left\{R(e_1 - e_2) - \mathrm{e}_0\, D\right\}\left(\hat{G}-G\right)g_{3e}\Big] \quad (43)$$

For $g_{3e} = 0$,

$$MSE\left(\hat{R}_{ge}\right) = E\left\{R^2\left(e_1 - e_2\right)^2 + R^2D^2e_0^2 - 2R^2De_0\left(e_1 - e_2\right)\right\}$$

$$= MSE\left(\hat{R}\right) + \Big[R^2D^2\left(\frac{1}{n} - \frac{1}{N}\right)C_0^2$$

$$-2R^2D\left(\frac{1}{n} - \frac{1}{N}\right)\left(\rho_{10}C_1C_0 - \rho_{20}C_2C_0\right)\Big]$$

$$= MSE\left(\hat{R}\right) + \left(\frac{1}{n} - \frac{1}{N}\right)\Big[C_0^2 R^2 D^2$$

$$-2C_0^2 R^2 D \left\{\rho_{10}\left(\frac{C_1}{C_0}\right) - \rho_{20}\left(\frac{C_2}{C_0}\right)\right\}\Big]$$

$$= MSE\left(\hat{R}\right) + \left(\frac{1}{n} - \frac{1}{N}\right)\left[C_0^2 R^2 D^2 - 2C_0^2 R^2 D^2\right]$$

$$MSE\left(\hat{R}_{ge}\right) = MSE(\hat{R}) - \left(\frac{1}{n} - \frac{1}{N}\right) C_0^2 R^2 D^2 \qquad (44)$$

which is equal to $MSE\left(\hat{R}_g\right)_{\min}$ in (23) if $g_{3e} = 0$. Thus, the estimator $\hat{R}_{ge} = g\left(\bar{y}_1, \bar{y}_2, \bar{x}, \hat{G}\right)$ attains the minimum mean square error $\hat{R}_g$ in (23) if $\hat{R}_{ge} = g\left(\bar{y}_1, \bar{y}_2, \bar{x}, \hat{G}\right)$ is based on estimated optimum $\hat{G}$ satisfies the following conditions at the point $P_e = \left(\bar{Y}_1, \bar{Y}_2, \bar{X}, G\right)$

$$\left.\begin{aligned}
& g\left(\bar{Y}_1, \bar{Y}_2, \bar{X}, G\right) = R \\
& g_{1e} = \left.\frac{\partial g\left(\bar{y}_1, \bar{y}_2, \bar{x}, \hat{G}\right)}{\partial \bar{y}_1}\right]_{\left(\bar{Y}_1, \bar{Y}_2, \bar{X}, G\right)} = \frac{1}{\bar{Y}_2} \\
& g_{2e} = \left.\frac{\partial g\left(\bar{y}_1, \bar{y}_2, \bar{x}, \hat{G}\right)}{\partial \bar{y}_2}\right]_{\left(\bar{Y}_1, \bar{Y}_2, \bar{X}, G\right)} = -\frac{\bar{Y}_1}{\bar{Y}_2^2} \\
& g_{0e} = \left.\frac{\partial g\left(\bar{y}_1, \bar{y}_2, \bar{x}, \hat{G}\right)}{\partial \bar{x}}\right]_{\left(\bar{Y}_1, \bar{Y}_2, \bar{X}, G\right)} = -\frac{RD}{\bar{X}} \\
& g_{3e} = \left.\frac{\partial g\left(\bar{y}_1, \bar{y}_2, \bar{x}, \hat{G}\right)}{\partial \hat{G}}\right]_{\left(\bar{Y}_1, \bar{Y}_2, \bar{X}, G\right)} = 0
\end{aligned}\right\} \qquad (45)$$

3. Some Special Cases Depending on Estimated Optimum

(a) First, we consider an estimator $\hat{R}_2$ of $\hat{R}_g$ to be

$$\hat{R}_2 = \frac{\bar{y}_1 + k\left(\bar{x} - \bar{X}\right)}{\bar{y}_2}$$

satisfying (i) $\left.\frac{\bar{y}_1 + k\left(\bar{x} - \bar{X}\right)}{\bar{y}_2}\right]_P = R$, (ii) $g_1 = \left.\frac{\partial \hat{R}_2}{\partial \bar{y}_1}\right]_P = \frac{1}{\bar{Y}_2}$,

(iii) $g_2 = \left.\frac{\partial \hat{R}_2}{\partial \bar{y}_2}\right]_P = -\frac{\bar{Y}_1}{\bar{Y}_2^2}$, (iv) $g_0 = \left.\frac{\partial \hat{R}_2}{\partial \bar{x}}\right]_P = \frac{k}{\bar{Y}_2} = G = -\frac{RD}{\bar{X}}$

and $\hat{R}_2$ attains the minimum value of mean square error of $\hat{R}_g$ to be

$$MSE\left(\hat{R}_2\right)_{\min} = MSE(\hat{R}) - \left(\frac{1}{n} - \frac{1}{N}\right) C_0^2 R^2 D^2 \tag{46}$$

of (23) for $k = \bar{Y}_2 G$ being unknown having some parameters lacks practical utility of $\hat{R}_2$. To overcome the practical utility of $\hat{R}_2$, we see that $\hat{R}_{2e}$ of $\hat{R}_{ge}$ depending on estimated optimum $g_{0e} = \frac{\hat{k}}{\bar{y}_2} = \hat{G}$ or $\hat{k} = \bar{y}_2 \hat{G}$ giving

$$\hat{R}_{2e} = \frac{\bar{y}_1 + \hat{k}\left(\bar{x} - \bar{X}\right)}{\bar{y}_2} = \frac{\bar{y}_1 + \bar{y}_2 \hat{G}\left(\bar{x} - \bar{X}\right)}{\bar{y}_2} \tag{47}$$

and satisfying (i) $\left.\frac{\bar{y}_1 + \bar{y}_2 \hat{G}\left(\bar{x} - \bar{X}\right)}{\bar{y}_2}\right]_{P_e} = R$, (ii) $\left.\frac{\partial \hat{R}_{2e}}{\partial \bar{y}_1}\right]_{P_e} = \frac{1}{\bar{Y}_2}$,

(iii) $\left.\frac{\partial \hat{R}_{2e}}{\partial \bar{y}_2}\right]_{P_e} = -\frac{\bar{Y}_1}{\bar{Y}_2^2}$, (iv) $\left.\frac{\partial \hat{R}_{2e}}{\partial \bar{x}}\right]_{P_e} = G = -\frac{RD}{\bar{X}}$

(v) $\left.\frac{\partial \hat{R}_{2e}}{\partial \hat{G}}\right]_{P_e} = 0$

of conditions in (45) attains the minimum value of mean square error of $\hat{R}_g$ given in (23) to be

$$MSE(\hat{R}) - \left(\frac{1}{n} - \frac{1}{N}\right) C_0^2 R^2 D^2 \tag{48}$$

(b) For $\hat{R}_3 = \frac{\bar{y}_1}{\bar{y}_2} \cdot \frac{\bar{x}}{\bar{\bar{X}}}$ and $\hat{R}_4 = \frac{\bar{y}_1}{\bar{y}_2} \cdot \frac{\bar{X}}{\bar{x}}$ {Singh (1965; 1967)}, the values of $g_0 = \frac{R}{\bar{\bar{X}}}$ for $\hat{R}_3$ and $g_0 = -\frac{R}{\bar{\bar{X}}}$ for $\hat{R}_4$ which when substituted in general expression of mean square error of $\hat{R}_g$ in (21) give

$$MSE\left(\hat{R}_3\right) = MSE(\hat{R}) + \left(\frac{1}{n} - \frac{1}{N}\right) C_0^2 \bar{X}^2 \left(\frac{R^2}{\bar{\bar{X}}^2} + 2\frac{R}{\bar{\bar{X}}} \cdot \frac{R}{\bar{\bar{X}}} D\right)$$

$$= MSE(\hat{R}) + \left(\frac{1}{n} - \frac{1}{N}\right) C_0^2 R^2 \left(1 + 2D\right) \qquad (49)$$

and $MSE\left(\hat{R}_4\right) = MSE(\hat{R}) + \left(\frac{1}{n} - \frac{1}{N}\right) C_0^2 \bar{X}^2 \left(\frac{R^2}{\bar{\bar{X}}^2} - 2\frac{R}{\bar{\bar{X}}} \cdot \frac{R}{\bar{\bar{X}}} D\right)$

$$= MSE(\hat{R}) + \left(\frac{1}{n} - \frac{1}{N}\right) C_0^2 R^2 \left(1 - 2D\right) \qquad (50)$$

The estimators $\hat{R}_3$ and $\hat{R}_4$ do not satisfy the minimizing conditions of (22) and (42); hence, $\hat{R}_3$ and $\hat{R}_4$ by (Singh, 1965; 1967) do not attain the minimum mean square error of (23) given by

$$MSE(\hat{R}) - \left(\frac{1}{n} - \frac{1}{N}\right) C_0^2 R^2 D^2 \qquad (51)$$

(c) For $\hat{R}_5$ and $\hat{R}_6$, finding the values of $g_0 = G$ and satisfying the conditions of $\hat{R}_g$ along with (22) which gives the optimum value of k and the estimated optimum $\hat{k}$ (based on sample observations)satisfying (42),we get the minimum mean square error of $\hat{R}_5$, $\hat{R}_6$, $\hat{R}_{5e}$ and $\hat{R}_{6e}$ {on the lines of (a) for $\hat{R}_2$ and $\hat{R}_{2e}$ } to be

$$MSE(\hat{R}) - \left(\frac{1}{n} - \frac{1}{N}\right) C_0^2 R^2 D^2 \qquad (52)$$

4. An Empirical Study

The theoretical results obtained in the study are illustrated here numerically, using the data given on page 177 of Singh and Chaudhary (2009). The data set is summarized as follows:

$\bar{Y}_1 = 856.4118$, $\bar{Y}_2 = 208.8824$, $\bar{X} = 199.4412$, $C_{y_1} = 0.8372$, $C_{y_2} = 0.7205$, $C_x = 0.7532$, $\rho_{y_1 y_2} = 0.2090$, $\rho_{y_1 x} = 0.2105$, $\rho_{y_2 x} = 0.9801$

Table 1: Percent Relative Efficiencies of the estimators with respect to $\hat{R}$

Estimators	Mean Square Error	Percent Relative Efficiency
$\hat{R}$	1.555166	100
$\hat{R}_1$	1.103932	141
$\hat{R}_2$	1.103932	141
$\hat{R}_3$	1.184017	131
$\hat{R}_4$	3.749341	41
$\hat{R}_5$	1.103932	141
$\hat{R}_6$	1.103932	141
$\hat{R}_g$	1.103932	141

From the above table we can easily conclude that the proposed generalized class of estimators has lesser mean square error than the conventional ratio estimator and the Singh (1965; 1967) estimators and is thus more efficient than the conventional ratio estimator and the Singh (1965; 1967) estimators.

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