Estimation of Population Median in Presence of Non - Response Under Two - Phase Sampling

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ABSTRACT

The present investigation deals with the problem of estimation of population median in presence of non-response under two-phase (double) sampling. Using information on two auxiliary variables, four general classes of estimators have been suggested for four different realistic situations of non – responses. It is shown that several estimators can be generated from our proposed classes of estimators. Proposed classes of estimators are compared with some contemporary estimators of population median under the similar realistic situations. The merits of the proposed strategies have been interpreted through empirical studies carried over three natural populations and one artificially generated population data sets. This establishes effectiveness of the suggested classes of estimators. Suitable recommendations to the survey statistician have been made.

Key wards: Median estimation, two-phase sampling, non response, study variable, auxiliary variable, bias, mean square error.

1. Introduction

In survey sampling, statisticians often come across the study of variables which have highly skewed distributions, such as income, expenditure etc. In such situations, the estimation of median deserves special attention. Kuk and Mak (1989) are the first to introduce the estimation of population median of the study variable using auxiliary information in survey sampling. Francisco and Fuller (1991) also dealt with the problem of estimation of the median as part of the estimation of a finite population distribution function. Later on Singh *et al*. (2001), Singh *et al*. (2006), Singh and Priyanka (2008) and Jhajj and Bhangu (2013) have contributed towards the improvement of estimation procedures of population median using information on one or two auxiliary variables.

It is worth to be mentioned that all the developments of estimation of population median are based on the complete response from the sampled units. However, no efforts have been made to estimate the population median in presence of non-response in the sampled units. Non response is one major problem, which is encountered by practitioners in the field of sample surveys. For example, in the case of income from milk yield surveys, the animal may be sold or may die during the survey period; in the case of vegetable or fruit surveys, the yield of some pickings may be damaged or lost or the enumerators may fail to record them. Thus, the observations may be missing for some of the time stages. Such non-response (missingness) can have different patterns and causes. In surveys covering human populations in most cases, information is not obtained from all the units in the survey at the first attempt even after some call-backs. For example selected families may not be home at the time of survey or do not cooperate with the interviewer even if contacted. This is particularly true in mail surveys in which questionnaires are mailed to the sampled respondents who are requested to send back their returns by some deadline. As many respondents do not reply, available sample of returns is incomplete. An estimate obtained from such incomplete data may be misleading especially when the respondents differ from the non-respondents. In order to reduce the effect of non-response in estimation of population mean, Hansen and Hurwitz (1946) gave a technique of sub-sampling of the non-responding group. Following Hansen and Hurwitz (1946) technique, several authors including Cochran (1977), Tripathi and Khare (1997), Tabasum and Khan (2004) and Singh and Kumar (2010 a, b) have contributed towards the improvement of the estimation procedures of population mean in presence of non-response using information on auxiliary variable. In many situations, information on the auxiliary variable may be readily available on all the units of the population; for example, tonnage (or seat capacity) of each vehicle or ship is known in survey sampling of transportation and number of beds in different hospitals may be known in hospital surveys. When such information is lacking, it is sometimes, relatively cheap to take a large preliminary sample in which auxiliary variable alone is measured. This technique is known as double sampling or two-phase sampling. Tabasum and Khan (2004) have mentioned that the procedure of double sampling can be applied in a household survey where the household size is used as an auxiliary variable for the estimation of family expenditure. Information can be obtained completely on the family size, while there may be non-response on the household expenditure.

Motivated with the above arguments and following the technique of sub-sampling of the non-responding group, we have proposed four general classes of estimators of population median for four different situations of non-response in two-phase sampling. The superiorities of the proposed classes of estimators over some contemporary median estimators under the similar realistic situations have been established through theoretical and empirical comparisons.

The next section discusses the formulation of the classes of estimators. Section 3 and 4 discusses their properties. Section 5 and 6 discusses the merits of the proposed strategies and section 7 concludes the paper.

2. Formulation of the Classes of Estimators

Consider a finite population $U = (U_1, U_2, U_3, \ldots, U_N)$ of N units. Let y, x and z are the variables under study, first auxiliary variable and second auxiliary variable respectively. Let y_k, x_k and z_k be the values of variables y, x and z respectively for the k-th (k = 1, 2, …, N) unit in the population. When the population median M_x of the auxiliary variable x is unknown, our purpose is to estimate the population median M_y of the study variable y from a sample obtained through a two-phase selection. Permitting simple random sampling without replacement (SRSWOR) design in each phase, the two-phase sampling scheme will be as follows:

i. The first phase sample $S_{n'}\left(S_{n'} \subset U\right)$ of size n' is drawn to observe the variable x only in order to furnish an estimate of M_x.

ii. The second phase sample $S_n\left(S_n \subset S_{n'}\right)$ of size n is drawn to observe the variable y only.

Assuming that the population median M_x of the auxiliary variable x is known, Kuk and Mak (1989) suggested a ratio estimator for the population median M_y of the study variable y as

$$t_1 = \hat{M}_y \frac{M_x}{\hat{M}_x} \tag{1}$$

where $\hat{M}_y$ and $\hat{M}_x$ are the sample estimators of M_y and M_x respectively based on a sample S_n of size n. Let $y_{(1)}, y_{(2)}, \ldots, y_{(n)}$ be the y values of sample units in ascending order. Further, let t be an integer such that $y_{(t)} \le M_y \le y_{(t+1)}$ and let $p = t/n$ be the proportion of y values in the sample that are less than or equal to the median value M_y, an unknown population parameter. If $\hat{p}$ is a predictor of p, the sample median $\hat{M}_y$ can be written in terms of quantities as $\hat{Q}(\hat{p})$ where $\hat{p} = 0.5$. For describing the estimator in equation (1), Kuk and Mak (1989) defined a matrix of proportions $\left(P_{ij}(x,y);\ i, j = 1, 2\right)$ of units in the population U as

	$y \le M_y$	$y > M_y$	Total
$x \le M_x$	$P_{11}(x,y)$	$P_{21}(x,y)$	$P_{\cdot 1}(x,y)$
$x > M_x$	$P_{12}(x,y)$	$P_{22}(x,y)$	$P_{\cdot 2}(x,y)$
Total	$P_{1\cdot}(x,y)$	$P_{2\cdot}(x,y)$	1

where $P_{11}(x,y)$ denotes the proportion of the units in the population with $y \le M_y$ and $x \le M_x$. In practice, the $P_{ij}(x,y)$ are usually unknown but can be estimated by $\hat{p}_{ij}(x,y)$ based on a similar cross-classification of the sample.

It may be noted that the estimator defined in equations (1) is based on prior knowledge of the population median M_x of the auxiliary variable x. In many situations of practical importance the population median M_x may not be known. Motivated with this point, Singh *et al.* (2001) discussed the problem of estimating the population median M_y in two-phase sampling and suggested ratio and regression type estimators of M_y as

$$t_1' = \hat{M}_y \frac{\hat{M}_x^1}{\hat{M}_x} \tag{2}$$

and

$$t_2' = \hat{M}_y + d\left(\hat{M}_x^1 - \hat{M}_x\right) \text{ respectively} \tag{3}$$

where $\hat{M}_x^1$ is sample median of the auxiliary variable x based on first phase sample $S_{n'}$ and d is a suitably chosen real constant such that the variance of the estimator t_2' is minimum. Sometimes even if population median M_x of the first auxiliary variable x is unknown, information on a second auxiliary variable z which is closely related to x may be used to estimate M_x from first phase sample $S_{n'}$. This type of situation, but in presence of complete response from the sampled units has been discussed among others by Singh *et al.* (2006).

Encouraged and fascinated with the above works, we have considered that at the first phase sample $S_{n'}$ of size n', all the units supplied information on the auxiliary variables x and z and at the second phase sample S_n of size n, let n_1 units supply information on y and n_2 units refuse to respond. Considering the non-response situations on the second phase sample, one may form an estimator by utilizing the information only from the respondents or take a sub-sample of the non-respondents and recontact them. Following Hansen and Hurwitz (1946) technique of sub-sampling the non-responding group adopted for estimation of population mean, a sub-sample of size m units $\left(m = n_2/k;\ k > 1\right)$ is selected at random (without replacement) from the n_2 non-respondent units and is enumerated by direct interview. It is assumed that response is obtained for all the m units and the whole population (i. e., U) is supposed to be consisting of two non-overlapping strata of N_1 and N_2 units. Stratum of N_1 responding units $\left(\text{denoted by } U_1\right)$ would respond on the first call at the second phase and the stratum of $N_2\left(N_2 = N - N_1\right)$ non-responding units $\left(\text{denoted by } U_2\right)$ would not respond on the first call at the second phase but will respond on the second call. Further, we assume that the strata sizes of N_1 and N_2 units are not known well in advance, see Tripathi and Khare (1997). The stratum weights of responding and non-responding groups are given by $W_1 = \frac{N_1}{N}$ and $W_2 = \frac{N_2}{N}$ and their estimates are considered as $\widehat{W}_1 = \frac{n_1}{n}$ and $\widehat{W}_2 = \frac{n_2}{n}$ respectively.

If non - response occurs on the study variable y as well as on the auxiliary variable x in the second phase sample, the estimators t_1' and t_2' may be considered in the following form as

$$t_1^* = \hat{M}_y^* \frac{\hat{M}_x^1}{\hat{M}_x^*} \tag{4}$$

and

$$t_2^* = \hat{M}_y^* + d^*\left(\hat{M}_x^1 - \hat{M}_x^*\right) \quad \text{respectively,} \tag{5}$$

where

d^* is a suitably chosen real constant such that the variance of the estimator t_2^* is minimum and $\hat{M}_y^*$ and $\hat{M}_x^*$ are the Hansen–Hurwitz type estimators for population medians M_y

and M_x respectively and are defined by $\hat{M}_y^* = \frac{n_1\hat{M}_{y_1} + n_2\hat{M}_{y_{2m}}}{n}$ and $\hat{M}_x^* = \frac{n_1\hat{M}_{x_1} + n_2\hat{M}_{x_{2m}}}{n}$ with $\left(\hat{M}_{x_1}, \hat{M}_{y_1}\right)$ and $\left(\hat{M}_{x_{2m}}, \hat{M}_{y_{2m}}\right)$ are the sample medians of (x, y) variables based on the samples of n_1 and m units respectively.

Similarly, when non-response situation is observed only on the study variable y, while the complete information on the auxiliary variable x is available in the second phase sample, the estimators t_1' and t_2' may be considered in the following form as

$$t_1^{**} = \hat{M}_y^* \frac{\hat{M}_x^1}{\hat{M}_x} \tag{6}$$

and

$$t_2^{**} = \hat{M}_y^* + d^{**}\left(\hat{M}_x^1 - \hat{M}_x\right) \text{ respectively,} \tag{7}$$

where d^{**} is a suitably chosen real constant such that the variance of the estimator t_2^{**} is minimum.

Motivated by the above suggestions and following the two-phase sampling structure defined above with the assumption that the population median M_x of the auxiliary variable x be unknown, we have proposed following four general classes of estimators of population median M_y of the study variable y applicable for four different situations of non-responses.

Situation I: In this case, we assume that the non- response conditions occur on the study variable y as well as on the auxiliary variable x in the second phase sample of size n and also the population median M_z of the second auxiliary variable z be known. Accordingly, we have suggested the general class of estimators of population median M_y in two-phase sampling as

$$T_1 = f\left(\hat{M}_y^*, \hat{M}_x^*, h_1\left(\hat{M}_x^1, \hat{M}_z^1\right)\right) \tag{8}$$

where $\hat{M}_z^1$ is the sample median of the variable z based on the first phase sample of size n' and $h_1\left(\hat{M}_x^1, \hat{M}_z^1\right)$ be a class of estimators of M_x using information on $\hat{M}_x^1$ and $\hat{M}_z^1$, such that

$$h_1\left(M_x, M_z\right) = M_x. \tag{9}$$

We treat the composite function $f\left(\hat{M}_y^*, \hat{M}_x^*, \hat{M}_x^1, \hat{M}_z^1\right)$ as one-to-one function of $\hat{M}_y^*, \hat{M}_x^*, \hat{M}_x^1$ and $\hat{M}_z^1$ denoted by $T_1 = F\left(\hat{M}_y^*, \hat{M}_x^*, \hat{M}_x^1, \hat{M}_z^1\right)$ such that

$$F\left(M_y, M_x, M_x, M_z\right) = M_y \Rightarrow \left.\frac{\partial F\left(\hat{M}_y^*, \hat{M}_x^*, \hat{M}_x^1, \hat{M}_z^1\right)}{\partial \hat{M}_y^*}\right|_{\left(M_y, M_x, M_x, M_z\right)} = 1 \tag{10}$$

where $\left(M_y, M_x, M_x, M_z\right)$ and $F\left(\hat{M}_y^*, \hat{M}_x^*, \hat{M}_x^1, \hat{M}_z^1\right)$ satisfies the following regularity conditions:

1. Whatever the chosen samples, $\left(\hat{M}_y^*, \hat{M}_x^*, \hat{M}_x^1, \hat{M}_z^1\right)$ assume values in a closed convex subspace, R^4 of the four dimensional real space containing the point $\left(M_y, M_x, M_x, M_z\right)$.

2. The function $F\left(\hat{M}_y^*, \hat{M}_x^*, \hat{M}_x^1, \hat{M}_z^1\right)$ is continuous and bounded in R^4.

3. The first, second and third order partial derivatives of $F\left(\hat{M}_y^*, \hat{M}_x^*, \hat{M}_x^1, \hat{M}_z^1\right)$ exist and are continuous and bounded in R^4.

It can be observed from equation (8) that the class of estimators T_1 is very wide in the sense for any parametric function, $f\left(\hat{M}_y^*, \hat{M}_x^*, h_1\left(\hat{M}_x^1, \hat{M}_z^1\right)\right)$ satisfying above regularity conditions with $F\left(M_y, M_x, M_x, M_z\right) = M_y$ can generate an estimators of M_y. For examples, the following ratio, product, regression and exponential type estimators are the member of the class T_1.

$$t_{1i} = \hat{M}_y^* \frac{\hat{M}_x'^i}{\hat{M}_x^*},\ t_{2i} = \hat{M}_y^* \frac{\hat{M}_x^*}{\hat{M}_x'^i},\ t_{3i} = \hat{M}_y^* + b_1\left(\hat{M}_x'^i - \hat{M}_x^*\right),\ t_{4i} = \hat{M}_y^* \exp\left(\frac{\hat{M}_x'^i - \hat{M}_x^*}{\hat{M}_x'^i + \hat{M}_x^*}\right);\ \left(i = 1, 2, \ldots, 4\right)$$

where

$$\hat{M}_x'^1 = \hat{M}_x^1 \frac{M_z}{\hat{M}_z^1},\ \hat{M}_x'^2 = \hat{M}_x^1 \frac{\hat{M}_z^1}{M_z},\ \hat{M}_x'^3 = \hat{M}_x^1 + b_2\left(M_z - \hat{M}_z^1\right),\ \hat{M}_x'^4 = \hat{M}_x^1 \exp\left(\frac{M_z - \hat{M}_z^1}{M_z + \hat{M}_z^1}\right)$$

and b_1 and b_2 are real constants.

Situation II: In this situation, we assume that the non-response occurs on the study variable y as well as on the auxiliary variables x and z in the second phase sample of size n and the population median M_z of the auxiliary variable z be unknown. Considering these aspects, we have formed the general class of estimators of M_y in two-phase sampling as

$$T_2 = g\left(\hat{M}_y^*, \hat{M}_x^*, \hat{M}_z^*, \hat{M}_x^1, \hat{M}_z^1\right) \tag{11}$$

where $g\left(\hat{M}_y^*, \hat{M}_x^*, \hat{M}_z^*, \hat{M}_x^1, \hat{M}_z^1\right)$ is a function of $\hat{M}_y^*, \hat{M}_x^*, \hat{M}_z^*, \hat{M}_x^1$ and $\hat{M}_z^1$ such that

$$g\left(M_y, M_x, M_z, M_x, M_z\right) = M_y \Rightarrow \left.\frac{\partial g\left(\hat{M}_y^*, \hat{M}_x^*, \hat{M}_z^*, \hat{M}_x^1, \hat{M}_z^1\right)}{\partial \hat{M}_y^*}\right|_{\left(M_y, M_x, M_z, M_x, M_z\right)} = 1 \tag{12}$$

where $\hat{M}_z^*$ is the Hansen–Hurwitz estimator for population median M_z and is defined by $\hat{M}_z^* = \frac{n_1\hat{M}_{z_1} + n_2\hat{M}_{z_{2m}}}{n}$ with $\hat{M}_{z_1}$ and $\hat{M}_{z_{2m}}$ are the sample medians of the variable z based on the samples of n_1 and m units respectively, $\left(M_y, M_x, M_z, M_x, M_z\right)$ and $g\left(\hat{M}_y^*, \hat{M}_x^*, \hat{M}_z^*, \hat{M}_x^1, \hat{M}_z^1\right)$ satisfies regularity conditions similar to those given for $\left(M_y, M_x, M_x, M_z\right)$ and $F\left(\hat{M}_y^*, \hat{M}_x^*, \hat{M}_x^1, \hat{M}_z^1\right)$ in equation (10).

It can be observed from equation (11) that for any parametric function, $g\left(\hat{M}_y^*, \hat{M}_x^*, \hat{M}_z^*, \hat{M}_x^1, \hat{M}_z^1\right)$ satisfying above regularity conditions with $g\left(M_y, M_x, M_z, M_x, M_z\right) = M_y$ for all M_y, can generate an estimator of M_y. For examples, we present below few ratio, product, regression and exponential type estimators as the members of the class of estimators T_2.

$$t'_{1i} = \hat{M}_y^* \frac{\hat{M}_x^1}{\hat{M}_x^{*i}},\ t'_{2i} = \hat{M}_y^* \frac{\hat{M}_x^{*i}}{\hat{M}_x^1},\ t'_{3i} = \hat{M}_y^* + b_3\left(\hat{M}_x^1 - \hat{M}_x^{*i}\right),\ t'_{4i} = \hat{M}_y^* \exp\left(\frac{\hat{M}_x^1 - \hat{M}_x^{*i}}{\hat{M}_x^1 + \hat{M}_x^{*i}}\right);\left(i = 1, 2, \ldots, 4\right),$$

$$t''_{1j}=\hat{M}_y^{*j}\frac{\hat{M}_x^1}{\hat{M}_x^*},\ t''_{2j}=\hat{M}_y^{*j}\frac{\hat{M}_x^1}{\hat{M}_x^*},\ t''_{3j}=\hat{M}_y^{*j}+b_1^*\left(\hat{M}_x^1-\hat{M}_x^*\right),\ t''_{4j}=\hat{M}_y^{*j}\exp\left(\frac{\hat{M}_x^1-\hat{M}_x^*}{\hat{M}_x^1+\hat{M}_x^*}\right);\ (j=1,2,\ldots,4)$$

where

$$\hat{M}_x^{*1}=\hat{M}_x^*\frac{\hat{M}_z^1}{\hat{M}_z^*},\ \hat{M}_x^{*2}=\hat{M}_x^*\frac{\hat{M}_z^*}{\hat{M}_z^1},\ \hat{M}_x^{*3}=\hat{M}_x^*+b_4\left(\hat{M}_z^1-\hat{M}_z^*\right),\ \hat{M}_x^{*4}=\hat{M}_x^*\exp\left(\frac{\hat{M}_z^1-\hat{M}_z^*}{\hat{M}_z^1+\hat{M}_z^*}\right),\ \hat{M}_y^{*1}=\hat{M}_y^*\frac{\hat{M}_z^1}{\hat{M}_z^*},$$

$$\hat{M}_y^{*2}=\hat{M}_y^*\frac{\hat{M}_z^*}{\hat{M}_z^1},\ \hat{M}_y^{*3}=\hat{M}_y^*+b_5\left(\hat{M}_z^1-\hat{M}_z^*\right)\text{ and }\hat{M}_y^{*4}=\hat{M}_y^*\exp\left(\frac{\hat{M}_z^1-\hat{M}_z^*}{\hat{M}_z^1+\hat{M}_z^*}\right),$$

b_1^*, b_3, b_4 and b_5 are real constants.

Situation III: In this case, we assume that the non-response situation occurs only on the study variable y while the complete information on the auxiliary variable x is available in second phase sample of size n and also the population median M_z of the second auxiliary variable z be known. Considering this situation, we have proposed the general class of estimators of population median M_y in two-phase sampling as

$$T_3=\varphi\left(\hat{M}_y^*,\hat{M}_x,h_1\left(\hat{M}_x^1,\hat{M}_z^1\right)\right) \tag{13}$$

We treat the composite function $\varphi\left(\hat{M}_y^*,\hat{M}_x,\hat{M}_x^1,\hat{M}_z^1\right)$ as one-to-one function of $\hat{M}_y^*,\hat{M}_x,\hat{M}_x^1$ and $\hat{M}_z^1$ denoted by $T_3=G\left(\hat{M}_y^*,\hat{M}_x,\hat{M}_x^1,\hat{M}_z^1\right)$ such that

$$G\left(M_y,M_x,M_x,M_z\right)=M_y\ \Rightarrow\ \left.\frac{\partial G\left(\hat{M}_y^*,\hat{M}_x,\hat{M}_x^1,\hat{M}_z^1\right)}{\partial\hat{M}_y^*}\right|_{\left(M_y,M_x,M_x,M_z\right)}=1 \tag{14}$$

where $\left(M_y,M_x,M_x,M_z\right)$ and $G\left(\hat{M}_y^*,\hat{M}_x,\hat{M}_x^1,\hat{M}_z^1\right)$ satisfies the similar regularity conditions given for $\left(M_y,M_x,M_x,M_z\right)$ and $F\left(\hat{M}_y^*,\hat{M}_x^*,\hat{M}_x^1,\hat{M}_z^1\right)$ in equation (10).

Situation IV: In this case, we assume that at the second phase sample non-response situation is found on the study variable y and the auxiliary variable z with unknown population mean M_z while the complete information about the auxiliary variable x is available there. Considering this situation, we have suggested the general class of estimators of population median M_y in two-phase sampling as

$$T_4=\psi\left(\hat{M}_y^*,\hat{M}_x,\hat{M}_z^*,\hat{M}_x^1,\hat{M}_z^1\right) \tag{15}$$

where $\psi\left(\hat{M}_y^*,\hat{M}_x,\hat{M}_z^*,\hat{M}_x^1,\hat{M}_z^1\right)$ is a function of $\hat{M}_y^*,\hat{M}_x,\hat{M}_z^*,\hat{M}_x^1$ and $\hat{M}_z^1$ such that

$$\psi\left(M_y,M_x,M_z,M_x,M_z\right)=M_y\ \Rightarrow\ \left.\frac{\partial\psi\left(\hat{M}_y^*,\hat{M}_x,\hat{M}_z^*,\hat{M}_x^1,\hat{M}_z^1\right)}{\partial\hat{M}_y^*}\right|_{\left(M_y,M_x,M_z,M_x,M_z\right)}=1 \tag{16}$$

where $\left(M_y,M_x,M_z,M_x,M_z\right)$ and $\psi\left(\hat{M}_y^*,\hat{M}_x,\hat{M}_z^*,\hat{M}_x^1,\hat{M}_z^1\right)$ satisfies the similar regularity conditions as presented for the class of estimators T_1.

Proceeding as above, it can be found that the classes of estimators T_3 and T_4 are also very wide and the following estimators can be identified as their member.

Estimators belonging to the class T_3 **:**

$$t^*_{1i} = \hat{M}^*_y \frac{\hat{M}'^i_x}{\hat{M}_x},\ t^*_{2i} = \hat{M}^*_y \frac{\hat{M}_x}{\hat{M}'^i_x},\ t^*_{2i} = \hat{M}^*_y + b_1\left(\hat{M}'^i_x - \hat{M}_x\right),\ t^*_{4i} = \hat{M}^*_y \exp\left(\frac{\hat{M}'^i_x - \hat{M}_x}{\hat{M}'^i_x + \hat{M}_x}\right);\ (i = 1, 2, \ldots, 4)$$

Estimators belonging to the class T_4 **:**

$$t^{**}_{1i} = \hat{M}^*_y \frac{\hat{M}^1_x}{\hat{M}''^i_x},\ t^{**}_{2i} = \hat{M}^*_y \frac{\hat{M}''^i_x}{\hat{M}^1_x},\ t^{**}_{3i} = \hat{M}^*_y + b_3\left(\hat{M}^1_x - \hat{M}''^i_x\right),\ t^{**}_{4i} = \hat{M}^*_y \exp\left(\frac{\hat{M}^1_x - \hat{M}''^i_x}{\hat{M}^1_x + \hat{M}''^i_x}\right);(i = 1, 2, \ldots, 4),$$

$$t^{***}_{1j} = \hat{M}^{*j}_y \frac{\hat{M}^1_x}{\hat{M}_x},\ t^{***}_{1j} = \hat{M}^{*j}_y \frac{\hat{M}^1_x}{\hat{M}_x},\ t^{***}_{1j} = \hat{M}^{*j}_y + b^*_1\left(\hat{M}^1_x - \hat{M}_x\right),\ t^{***}_{1j} = \hat{M}^{*j}_y \exp\left(\frac{\hat{M}^1_x - \hat{M}_x}{\hat{M}^1_x + \hat{M}_x}\right);(j = 1, 2, \ldots, 4) \text{ where}$$

$$\hat{M}''^1_x = \hat{M}_x \frac{\hat{M}^1_z}{\hat{M}^*_z},\ \hat{M}''^2_x = \hat{M}_x \frac{\hat{M}^*_z}{\hat{M}^1_z},\ \hat{M}''^3_x = \hat{M}_x + b_4\left(\hat{M}^1_z - \hat{M}^*_z\right) \text{ and } \hat{M}''^4_x = \hat{M}_x \exp\left(\frac{\hat{M}^1_z - \hat{M}^*_z}{\hat{M}^1_z + \hat{M}^*_z}\right).$$

3. Bias and Mean Square Errors of the Proposed Classes of Estimators $T_i\left(i = 1, 2, \ldots, 4\right)$

The bias and mean square errors (M. S. E.s) of the proposed classes of estimators $T_i\left(i = 1, 2, \ldots, 4\right)$ to the first order of approximations are derived under large sample approximations using the following transformations:

$$\hat{M}^*_y = M_y\left(1+e_0\right),\ \hat{M}^*_x = M_x\left(1+e_1\right),\ \hat{M}^*_z = M_z\left(1+e_2\right),\ \hat{M}_x = M_x\left(1+e_3\right),\ \hat{M}^1_x = M_x\left(1+e'_1\right),\ \hat{M}^1_z = M_z\left(1+e'_2\right).$$

Such that $|e_i|$ and $|e'_j|$ are < 1 (i = 0, 1, . . ., 3; j= 1, 2).

Now, we define following two matrices $\left\{P_{ij}(y,z) \text{ and } P_{ij}(x,z);\ i, j = 1, 2\right\}$ of the units of the population U as

	$z \le M_z$	$z > M_z$	Total
$x \le M_x$	$P_{11}(x,z)$	$P_{21}(x,z)$	$P_{\cdot 1}(x,z)$
$x > M_x$	$P_{12}(x,z)$	$P_{22}(x,z)$	$P_{\cdot 2}(x,z)$
Total	$P_{1\cdot}(x,z)$	$P_{2\cdot}(x,z)$	1

and

	$z \le M_z$	$z > M_z$	Total
$y \le M_y$	$P_{11}(y,z)$	$P_{21}(y,z)$	$P_{\cdot 1}(y,z)$
$y > M_y$	$P_{12}(y,z)$	$P_{22}(y,z)$	$P_{\cdot 2}(y,z)$
Total	$P_{1\cdot}(y,z)$	$P_{2\cdot}(y,z)$	1

The matrix of proportions $\left\{P^*_{ij}(x,y),\ P^*_{ij}(y,z) \text{ and } P^*_{ij}(x,z);\ i, j = 1, 2\right\}$ of units in the population U_2 (non-responding group of the population U) can be defined in the same way as defined for the matrix proportions $\left\{P_{ij}(x,y),\ P_{ij}(y,z) \text{ and } P_{ij}(x,z);\ i, j = 1, 2\right\}$.

It is to be noted that $\rho_{xy} = 4P_{11}(x,y) - 1$ is the correlation coefficient between the variables x and y based on the population U, goes from -1 to 1 as $P_{11}(x,y)$ increases from 0 to $1/2$. Similarly, $\rho_{xz} = 4P_{11}(x,z) - 1$ and $\rho_{yz} = 4P_{11}(y,z) - 1$ are the correlation coefficients between the respective variables based on the population U and ($\rho_{xy(2)} = 4P^{*}_{11}(x,y) - 1$, $\rho_{xz(2)} = 4P^{*}_{11}(x,z) - 1$, $\rho_{yz(2)} = 4P^{*}_{11}(y,z) - 1$) are the correlation coefficients between the respective variables based on the non-responding part (U_2) of the population.

Further, we have the following expectations:

$$\left.\begin{aligned} &E\left(e_0^2\right) = f_1 \frac{\left\{M_y f_y(M_y)\right\}^{-2}}{4} + W_2 \frac{(k-1)}{n} \frac{\left\{M_{y_2} f_{y_2}(M_{y_2})\right\}^{-2}}{4},\ f_1 = \left(\frac{1}{n} - \frac{1}{N}\right), \\ &E\left(e_1^2\right) = f_1 \frac{\left\{M_x f_x(M_x)\right\}^{-2}}{4} + W_2 \frac{(k-1)}{n} \frac{\left\{M_{x_2} f_{x_2}(M_{x_2})\right\}^{-2}}{4},\ f_2 = \left(\frac{1}{n'} - \frac{1}{N}\right), \\ &E\left(e_2^2\right) = f_1 \frac{\left\{M_z f_z(M_z)\right\}^{-2}}{4} + W_2 \frac{(k-1)}{n} \frac{\left\{M_{z_2} f_{z_2}(M_{z_2})\right\}^{-2}}{4},\ E\left(e_1'^2\right) = f_2 \frac{\left\{M_x f_x(M_x)\right\}^{-2}}{4}, \end{aligned}\right\} \quad (17)$$

$$\left.\begin{aligned} &E\left(e_0 e_1\right) = f_1 \frac{\rho_{xy}\left\{M_x M_y f_x(M_x) f_y(M_y)\right\}^{-1}}{4} + W_2 \frac{(k-1)}{n} \frac{\rho_{xy(2)}\left\{M_{x_2} M_{y_2} f_{x_2}(M_{x_2}) f_{y_2}(M_{y_2})\right\}^{-1}}{4} \\ &E\left(e_0 e_2\right) = f_1 \frac{\rho_{yz}\left\{M_y M_z f_y(M_y) f_z(M_z)\right\}^{-1}}{4} + W_2 \frac{(k-1)}{n} \frac{\rho_{yz(2)}\left\{M_{y_2} M_{z_2} f_{y_2}(M_{y_2}) f_{z_2}(M_{z_2})\right\}^{-1}}{4} \\ &E\left(e_1 e_2\right) = f_1 \frac{\rho_{xz}\left\{M_x M_z f_x(M_x) f_z(M_z)\right\}^{-1}}{4} + W_2 \frac{(k-1)}{n} \frac{\rho_{xz(2)}\left\{M_{x_2} M_{z_2} f_{x_2}(M_{x_2}) f_{z_2}(M_{z_2})\right\}^{-1}}{4} \\ &E\left(e_0 e_1'\right) = f_2 \frac{\rho_{xy}\left\{M_x M_y f_x(M_x) f_y(M_y)\right\}^{-1}}{4},\ E\left(e_0 e_3\right) = f_1 \frac{\rho_{xy}\left\{M_x M_y f_x(M_x) f_y(M_y)\right\}^{-1}}{4}, \\ &E\left(e_0 e_2'\right) = f_2 \frac{\rho_{yz}\left\{M_y M_z f_y(M_y) f_z(M_z)\right\}^{-1}}{4},\ E\left(e_2 e_3\right) = f_1 \frac{\rho_{xz}\left\{M_x M_z f_x(M_x) f_z(M_z)\right\}^{-1}}{4}, \\ &f_3 = \left(\frac{1}{n} - \frac{1}{n'}\right), E\left(e_1' e_3\right) = E\left(e_1 e_1'\right) = f_2 \frac{\left\{M_x f_x(M_x)\right\}^{-2}}{4},\ E\left(e_2'^2\right) = f_2 \frac{\left\{M_z f_z(M_z)\right\}^{-2}}{4}, \\ &E\left(e_1 e_2'\right) = E\left(e_1' e_2\right) = E\left(e_1' e_2'\right) = E\left(e_2' e_3\right) = f_2 \frac{\rho_{xz}\left\{M_x M_z f_x(M_x) f_z(M_z)\right\}^{-1}}{4}, \\ &E\left(e_3^2\right) = E\left(e_1 e_3\right) = f_1 \frac{\left\{M_x f_x(M_x)\right\}^{-2}}{4},\ E\left(e_0\right) = E\left(e_1\right) = E\left(e_2\right) = E\left(e_3\right) = E\left(e_1'\right) = E\left(e_2'\right) = 0, \end{aligned}\right\}$$

where it is assumed that as $N \rightarrow \infty$ (obviously then N_1 and N_2 both $\rightarrow \infty$), the distribution of the variables (x, y, z) approaches a continuous distribution with marginal densities $\left\{f_x(x), f_y(y), f_z(z)\right\}$ and $\left\{f_{x_2}(x_2), f_{y_2}(y_2), f_{z_2}(z_2)\right\}$ for the whole population U and the non-responding part of the population (i. e. U_2) respectively. This assumption holds in particular under a super population model framework, treating the values of (x, y, z) in the population as a realization of N independent observations from a continuous distribution. It is assumed that $f_x(x)$, $f_y(y)$, $f_z(z)$, $f_{x_2}(x_2)$, $f_{y_2}(y_2)$

and $f_{z_2}(z_2)$ are positive. It may be noted that under these conditions, the sample median $\hat{M}_y$ is consistent and asymptotically normal (Gross, 1980).

Now, to express T_1 in terms of e's, we expand $F\left(\hat{M}_y^*, \hat{M}_x^*, \hat{M}_x^1, \hat{M}_z^1\right)$ about the point $\left(M_y, M_x, M_x, M_z\right)$ in a third order Taylor's series and we have

$$\begin{aligned}
F\left(\hat{M}_y^*, \hat{M}_x^*, \hat{M}_x^1, \hat{M}_z^1\right) &= F\left(M_y, M_x, M_x, M_z\right) + d_1\left(\hat{M}_y^* - M_y\right) + d_2\left(\hat{M}_x^* - M_x\right) + d_3\left(\hat{M}_x^1 - M_x\right) + d_4\left(\hat{M}_z^1 - M_z\right) \\
&+ \frac{1}{2}\Big\{d_{11}\left(\hat{M}_y^* - M_y\right)^2 + d_{22}\left(\hat{M}_x^* - M_x\right)^2 + d_{33}\left(\hat{M}_x^1 - M_x\right)^2 + d_{44}\left(\hat{M}_z^1 - M_z\right)^2 \\
&+ 2d_{12}\left(\hat{M}_y^* - M_y\right)\left(\hat{M}_x^* - M_x\right) + 2d_{13}\left(\hat{M}_y^* - M_y\right)\left(\hat{M}_x^1 - M_x\right) + 2d_{14}\left(\hat{M}_y^* - M_y\right)\left(\hat{M}_z^1 - M_z\right) \\
&+ 2d_{23}\left(\hat{M}_x^* - M_x\right)\left(\hat{M}_x^1 - M_x\right) + 2d_{24}\left(\hat{M}_x^* - M_x\right)\left(\hat{M}_z^1 - M_z\right) + 2d_{34}\left(\hat{M}_x^1 - M_x\right)\left(\hat{M}_z^1 - M_z\right)\Big\} \\
&+ \frac{1}{6}\left\{\left(\hat{M}_y^* - M_y\right)\frac{\partial}{\partial \hat{M}_y^*} + \left(\hat{M}_x^* - M_x\right)\frac{\partial}{\partial \hat{M}_x^*} + \left(\hat{M}_x^1 - M_x\right)\frac{\partial}{\partial \hat{M}_x^1} + \left(\hat{M}_z^1 - M_z\right)\frac{\partial}{\partial \hat{M}_z^1}\right\}^3 F\left(\hat{M}_{y\theta}^*, \hat{M}_{x\theta}^*, \hat{M}_{x\theta}^1, \hat{M}_{z\theta}^1\right)
\end{aligned}$$

where

$$d_1 = \left.\frac{\partial F\left(\hat{M}_y^*, \hat{M}_x^*, \hat{M}_x^1, \hat{M}_z^1\right)}{\partial \hat{M}_y^*}\right|_{\left(M_y, M_x, M_x, M_z\right)}, \quad d_2 = \left.\frac{\partial F\left(\hat{M}_y^*, \hat{M}_x^*, \hat{M}_x^1, \hat{M}_z^1\right)}{\partial \hat{M}_x^*}\right|_{\left(M_y, M_x, M_x, M_z\right)},$$

$$d_3 = \left.\frac{\partial F\left(\hat{M}_y^*, \hat{M}_x^*, \hat{M}_x^1, \hat{M}_z^1\right)}{\partial \hat{M}_x^1}\right|_{\left(M_y, M_x, M_x, M_z\right)}, \quad d_4 = \left.\frac{\partial F\left(\hat{M}_y^*, \hat{M}_x^*, \hat{M}_x^1, \hat{M}_z^1\right)}{\partial \hat{M}_z^1}\right|_{\left(M_y, M_x, M_x, M_z\right)}$$

$\left(d_{11}, d_{22}, d_{33}, d_{44}, d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, d_{34}\right)$ are the second order partial derivatives of $F\left(\hat{M}_y^*, \hat{M}_x^*, \hat{M}_x^1, \hat{M}_z^1\right)$ at the point $\left(M_y, M_x, M_x, M_z\right)$ and $\hat{M}_{y\theta}^* = M_y + \theta\left(\hat{M}_y^* - M_y\right)$, $\hat{M}_{x\theta}^* = M_x + \theta\left(\hat{M}_x^* - M_x\right)$, $\hat{M}_{x\theta}^1 = M_x + \theta\left(\hat{M}_x^1 - M_x\right)$, $\hat{M}_{z\theta}^1 = M_z + \theta\left(\hat{M}_z^1 - M_z\right)$ for $\left(0 < \theta < 1\right)$.

In the light of the conditions mentioned for $F\left(\hat{M}_y^*, \hat{M}_x^*, \hat{M}_x^1, \hat{M}_z^1\right)$ in equation (10), it is noted that

$$F\left(M_y, M_x, M_x, M_z\right) = M_y \Rightarrow d_1 = 1 \text{ and } d_{11} = \left.\frac{\partial^2 F\left(\hat{M}_y^*, \hat{M}_x^*, \hat{M}_x^1, \hat{M}_z^1\right)}{\partial \hat{M}_y^{*2}}\right|_{\left(M_y, M_x, M_x, M_z\right)} = 0. \tag{18}$$

Since the population median M_x of the auxiliary variable x is unknown, we have to impose the constraint as

$$d_2 = -d_3. \tag{19}$$

Thus, expressing $F\left(\hat{M}_y^*, \hat{M}_x^*, \hat{M}_x^1, \hat{M}_z^1\right)$ in terms of e's and neglecting the terms of e's having power greater than two we have the expansion of T_1 as

$$T_1 = F\left(\hat{M}_y^*, \hat{M}_x^*, \hat{M}_x^1, \hat{M}_z^1\right) = M_y\left(1+e_0\right) + d_2 M_x\left(e_1 - e_1'\right) + d_4 M_z e_2' + \frac{1}{2}\left\{\begin{array}{l} M_x^2\left(d_{22}e_1^2 + d_{33}e_1'^2 + 2d_{23}e_1e_1'\right) + d_{44}M_z^2 e_2'^2 + 2M_yM_x\left(d_{12}e_0e_1 + d_{13}e_0e_1'\right) \\ +2d_{14}M_yM_ze_0e_2' + 2M_xM_z\left(d_{24}e_1e_2' + d_{34}e_1'e_2'\right) \end{array}\right\} \quad (20)$$

Similarly, expressing $T_i\left(i = 2, 3, 4\right)$ in terms of e's we have

$$T_2 = g\left(\hat{M}_y^*, \hat{M}_x^*, \hat{M}_z^*, \hat{M}_x^1, \hat{M}_z^1\right) = M_y\left(1+e_0\right) + c_2 M_x\left(e_1 - e_1'\right) + c_3 M_z\left(e_2 - e_2'\right) \quad (21)$$

$$+\frac{1}{2}\left\{\begin{array}{l} M_x^2\left(c_{22}e_1^2 + c_{44}e_1'^2 + 2c_{24}e_1e_1'\right) + M_z^2\left(c_{33}e_2^2 + c_{55}e_2'^2 + 2c_{35}e_2e_2'\right) + 2M_yM_x\left(c_{12}e_0e_1 + c_{14}e_0e_1'\right) \\ +2M_yM_z\left(c_{13}e_0e_2 + c_{15}e_0e_2'\right) + 2M_xM_z\left(c_{23}e_1e_2 + c_{25}e_1e_2' + c_{34}e_1'e_2 + c_{45}e_1'e_2'\right) \end{array}\right\}$$

$$T_3 = G\left(\hat{M}_y^*, \hat{M}_x, \hat{M}_x^1, \hat{M}_z^1\right) = M_y\left(1+e_0\right) + p_2 M_x\left(e_3 - e_1'\right) + p_4 M_z e_2' + \frac{1}{2}\left\{\begin{array}{l} M_x^2\left(p_{22}e_3^2 + p_{33}e_1'^2 + 2p_{23}e_1'e_3\right) + M_z^2 p_{44}e_2'^2 + 2M_yM_x\left(p_{12}e_0e_3 + p_{13}e_0e_1'\right) \\ +2p_{14}M_yM_ze_0e_2' + 2M_xM_z\left(p_{24}e_2'e_3 + p_{34}e_1'e_2'\right) \end{array}\right\} \quad (22)$$

and

$$T_4 = \psi\left(\hat{M}_y^*, \hat{M}_x, \hat{M}_z^*, \hat{M}_x^1, \hat{M}_z^1\right) = M_y\left(1+e_0\right) + q_2 M_x\left(e_3 - e_1'\right) + q_3 M_z\left(e_2 - e_2'\right) + \frac{1}{2}\left\{\begin{array}{l} M_x^2\left(q_{22}e_3^2 + q_{44}e_1'^2 + 2q_{24}e_1'e_3\right) + M_z^2\left(q_{33}e_2^2 + q_{55}e_2'^2 + 2q_{35}e_2e_2'\right) \\ +2M_yM_x\left(q_{12}e_0e_3 + q_{14}e_0e_1'\right) + 2M_yM_z\left(q_{13}e_0e_2 + q_{15}e_0e_2'\right) \\ +2M_xM_z\left(q_{23}e_2e_3 + q_{25}e_2'e_3 + q_{34}e_1'e_2 + q_{45}e_1'e_2'\right) \end{array}\right\} \quad (23)$$

where

$$c_2 = \left.\frac{\partial g\left(\hat{M}_y^*, \hat{M}_x^*, \hat{M}_z^*, \hat{M}_x^1, \hat{M}_z^1\right)}{\partial \hat{M}_x^*}\right|_{\left(M_y, M_x, M_z, M_x, M_z\right)} = -\left.\frac{\partial g\left(\hat{M}_y^*, \hat{M}_x^*, \hat{M}_z^*, \hat{M}_x^1, \hat{M}_z^1\right)}{\partial \hat{M}_x^1}\right|_{\left(M_y, M_x, M_z, M_x, M_z\right)} \quad \left\{\text{as } M_x \text{ is unknown}\right\},$$

$$c_3 = \left.\frac{\partial g\left(\hat{M}_y^*, \hat{M}_x^*, \hat{M}_z^*, \hat{M}_x^1, \hat{M}_z^1\right)}{\partial \hat{M}_z^*}\right|_{\left(M_y, M_x, M_z, M_x, M_z\right)} = -\left.\frac{\partial g\left(\hat{M}_y^*, \hat{M}_x^*, \hat{M}_z^*, \hat{M}_x^1, \hat{M}_z^1\right)}{\partial \hat{M}_z^1}\right|_{\left(M_y, M_x, M_z, M_x, M_z\right)} \quad \left\{\text{as } M_z \text{ is unknown}\right\},$$

$$p_2 = \left.\frac{\partial}{\partial \hat{M}_x} G\left(\hat{M}_y^*, \hat{M}_x, \hat{M}_x^1, \hat{M}_z^1\right)\right|_{\left(M_y, M_x, M_x, M_z\right)} = -\left.\frac{\partial}{\partial \hat{M}_x^1} G\left(\hat{M}_y^*, \hat{M}_x, \hat{M}_x^1, \hat{M}_z^1\right)\right|_{\left(M_y, M_x, M_x, M_z\right)} \quad \left\{\text{as } M_x \text{ is unknown}\right\},$$

$$q_2 = \left.\frac{\partial \psi\left(\hat{M}_y^*, \hat{M}_x, \hat{M}_z^*, \hat{M}_x^1, \hat{M}_z^1\right)}{\partial \hat{M}_x^*}\right|_{\left(M_y, M_x, M_z, M_x, M_z\right)} = -\left.\frac{\partial \psi\left(\hat{M}_y^*, \hat{M}_x, \hat{M}_z^*, \hat{M}_x^1, \hat{M}_z^1\right)}{\partial \hat{M}_x^1}\right|_{\left(M_y, M_x, M_z, M_x, M_z\right)} \quad \left\{\text{as } M_x \text{ is unknown}\right\},$$

$$q_3 = \left.\frac{\partial \psi\left(\hat{M}_y^*, \hat{M}_x, \hat{M}_z^*, \hat{M}_x^1, \hat{M}_z^1\right)}{\partial \hat{M}_z^*}\right|_{\left(M_y, M_x, M_z, M_x, M_z\right)} = -\left.\frac{\partial \psi\left(\hat{M}_y^*, \hat{M}_x, \hat{M}_z^*, \hat{M}_x^1, \hat{M}_z^1\right)}{\partial \hat{M}_z^1}\right|_{\left(M_y, M_x, M_z, M_x, M_z\right)} \quad \left\{\text{as } M_z \text{ is unknown}\right\},$$

$$p_4 = \left.\frac{\partial}{\partial \hat{M}_z^1} G\left(\hat{M}_y^*, \hat{M}_x, \hat{M}_x^1, \hat{M}_z^1\right)\right|_{\left(M_y, M_x, M_x, M_z\right)} \quad \text{and}$$

$(c_{22}, c_{33}, c_{44}, c_{55}, c_{12}, c_{13}, c_{14}, c_{15}, c_{23}, c_{24}, c_{25}, c_{34}, c_{35}, c_{45})$ are the second order partial derivatives of $g\left(\hat{M}_y^*, \hat{M}_x^*, \hat{M}_z^*, \hat{M}_x^1, \hat{M}_z^1\right)$ at the point $\left(M_y, M_x, M_z, M_x, M_z\right)$,

$\left(p_{22}, p_{33}, p_{44}, p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}\right)$ are the second order partial derivatives of $G\left(\hat{M}_y^*, \hat{M}_x, \hat{M}_x^1, \hat{M}_z^1\right)$ at the point $\left(M_y, M_x, M_x, M_z\right)$ and

$(q_{22}, q_{33}, q_{44}, q_{55}, q_{12}, q_{13}, q_{14}, q_{15}, q_{23}, q_{24}, q_{25}, q_{34}, q_{35}, q_{45})$ are the second order partial derivatives of $\psi\left(\hat{M}_y^*, \hat{M}_x, \hat{M}_z^*, \hat{M}_x^1, \hat{M}_z^1\right)$ at the point $\left(M_y, M_x, M_z, M_x, M_z\right)$.

Taking expectations on both sides of the equations (20) - (23) and using the results from equation (17), we obtain the expressions for bias B(.) and mean square errors M(.) of the classes of estimators $T_i (i = 1, 2, \ldots, 4)$ to the first order of approximations as

$$B\left(T_1\right) = E\left(T_1 - M_y\right) \tag{24}$$

$$= \frac{M_x^2}{2}\left\{d_{22}A' + \frac{\left(M_x f_x(M_x)\right)^{-2}}{4} f_2\left(d_{33} + 2d_{23}\right)\right\} + d_{44} f_2 \frac{M_z^2\left(M_z f_z(M_z)\right)^{-2}}{8} + d_{14} f_2 M_y M_z \rho_{yz} \frac{\left(M_y M_z f_y(M_y) f_z(M_z)\right)^{-1}}{4}$$
$$+ M_x M_y\left\{d_{12}C' + d_{13} f_2 \rho_{xy} \frac{\left(M_x M_y f_x(M_x) f_y(M_y)\right)^{-1}}{4}\right\} + f_2 M_x M_z \rho_{xz}\left\{\frac{\left(M_x M_z f_x(M_x) f_z(M_z)\right)^{-1}}{4}\right\}\left(d_{24} + d_{34}\right),$$

$$B\left(T_2\right) = E\left(T_2 - M_y\right)$$
$$= \frac{M_x^2}{2}\left\{c_{22}A' + f_2 \frac{\left(M_x f_x(M_x)\right)^{-2}}{4}\left(c_{44} + 2c_{24}\right)\right\} + \frac{M_z^2}{2}\left\{c_{33}B' + f_2 \frac{\left(M_z f_z(M_z)\right)^{-2}}{4}\left(c_{55} + 2c_{35}\right)\right\} \tag{25}$$
$$+ M_x M_y\left\{c_{12}C' + c_{14} f_2 \rho_{xy} \frac{\left\{\left(M_x M_y f_x(M_x) f_y(M_y)\right\}^{-1}}{4}\right\} + M_y M_z\left\{c_{13}D' + c_{15} f_2 \rho_{yz} \frac{\left\{\left(M_y M_z f_y(M_y) f_z(M_z)\right\}^{-1}}{4}\right\}$$
$$+ M_x M_z\left\{c_{23}E' + f_2 \rho_{xz} \frac{\left\{\left(M_x M_z f_x(M_x) f_z(M_z)\right\}^{-1}}{4}\left(c_{25} + c_{34} + c_{45}\right)\right\}$$

$$B\left(T_3\right) = E\left(T_3 - M_y\right)$$
$$= \frac{M_x^2}{2}\left\{\frac{\left(M_x f_x(M_x)\right)^{-2}}{4}\right\}\left(p_{22} f_1 + p_{33} f_2 + 2p_{23} f_2\right) + \frac{M_z^2}{2} p_{44} f_2 \frac{\left(M_z f_z(M_z)\right)^{-2}}{4} + p_{14} f_2 M_y M_z \rho_{yz} \frac{\left(M_y M_z f_y(M_y) f_z(M_z)\right)^{-1}}{4}$$
$$+ M_y M_x\left\{\left(p_{12} f_1 + p_{13} f_2\right)\rho_{xy} \frac{\left(M_x M_y f_x(M_x) f_y(M_y)\right)^{-1}}{4}\right\} + M_x M_z \rho_{xz} f_2 \frac{\left(M_x M_z f_x(M_x) f_z(M_z)\right)^{-1}}{4}\left(p_{24} + p_{34}\right), \tag{26}$$

$$B(T_4) = E(T_4 - M_y)$$
$$= \frac{M_x^2}{2}\left\{\frac{\left(M_x f_x(M_x)\right)^{-2}}{4}\left(q_{22}f_1 + f_2 q_{44} + 2f_2 q_{24}\right)\right\} + \frac{M_z^2}{2}\left\{q_{33}B' + f_2\frac{\left(M_z f_z(M_z)\right)^{-2}}{4}\left(q_{55} + 2q_{35}\right)\right\} \quad (27)$$

$$+M_y M_x \rho_{xy}\frac{\left(M_x M_y f_x(M_x) f_y(M_y)\right)^{-1}}{4}\left(q_{12}f_1 + q_{14}f_2\right) + M_y M_z\left\{q_{13}D' + q_{15}f_2\rho_{yz}\frac{\left(M_y M_z f_y(M_y) f_z(M_z)\right)^{-1}}{4}\right\}$$
$$+M_x M_z \rho_{xz}\frac{\left(M_x M_z f_x(M_x) f_z(M_z)\right)^{-1}}{4}\left\{q_{23}f_1 + f_2\left(q_{25} + q_{34} + q_{45}\right)\right\},$$

$$M(T_1) = E\left(T_1 - M_y\right)^2 = M_y^2\left\{f_1\frac{\left\{M_y f_y(M_y)\right\}^{-2}}{4} + W_2\frac{(k-1)}{n}\frac{\left\{M_{y_2} f_{y_2}(M_{y_2})\right\}^{-2}}{4}\right\} + d_2^2 M_x^2 A + d_4^2 M_z^2 f_2\frac{\left\{M_z f_z(M_z)\right\}^{-2}}{4}$$
$$+ 2d_2 M_y M_x C + d_4 M_y M_z f_2\frac{\rho_{yz}}{2}\left\{M_y M_z f_y(M_y) f_z(M_z)\right\}^{-1} \quad (28)$$

$$M(T_2) = E\left(T_2 - M_y\right)^2 = M_y^2\left\{f_1\frac{\left\{M_y f_y(M_y)\right\}^{-2}}{4} + W_2\frac{(k-1)}{n}\frac{\left\{M_{y_2} f_{y_2}(M_{y_2})\right\}^{-2}}{4}\right\} + c_2^2 M_x^2 A + c_3^2 M_z^2 B + 2c_2 M_y M_x C$$
$$+ 2c_3 M_y M_z D + 2c_2 c_3 M_x M_z E, \quad (29)$$

$$M(T_3) = E\left(T_3 - M_y\right)^2$$
$$= M_y^2\left\{f_1\frac{\left\{M_y f_y(M_y)\right\}^{-2}}{4} + W_2\frac{(k-1)}{n}\frac{\left\{M_{y_2} f_{y_2}(M_{y_2})\right\}^{-2}}{4}\right\} + p_2^2 f_3 M_x^2\frac{\left\{M_x f_x(M_x)\right\}^{-2}}{4} + p_4^2 f_2 M_z^2\frac{\left\{M_z f_z(M_z)\right\}^{-2}}{4}$$
$$+ p_2 f_3 M_y M_x\frac{\rho_{xy}\left\{M_x M_y f_x(M_x) f_y(M_y)\right\}^{-1}}{2} + p_4 f_2 M_y M_Z\frac{\rho_{yz}\left\{M_y M_z f_y(M_y) f_z(M_z)\right\}^{-1}}{2} \quad (30)$$

and

$$M(T_4) = E\left(T_4 - M_y\right)^2$$
$$= M_y^2\left\{f_1\frac{\left\{M_y f_y(M_y)\right\}^{-2}}{4} + W_2\frac{(k-1)}{n}\frac{\left\{M_{y_2} f_{y_2}(M_{y_2})\right\}^{-2}}{4}\right\} + q_2^2 f_3 M_x^2\frac{\left\{M_x f_x(M_x)\right\}^{-2}}{4} + q_3^2 M_z^2 B + 2q_3 M_y M_z D$$
$$+ q_2 f_3 M_y M_x\frac{\rho_{xy}\left\{M_y M_x f_y(M_y) f_x(M_x)\right\}^{-1}}{2} + q_2 q_3 M_x M_z f_3\frac{\rho_{xz}\left\{M_x M_z f_x(M_x) f_z(M_z)\right\}^{-1}}{2} \quad (31)$$

where

M_{y_2}, M_{x_2} and M_{z_2} are the population medians of the variables y, x and z respectively based on the non-responding part U_2,

$$A= f_3\frac{\left\{M_xf_x(M_x)\right\}^{-2}}{4}+W_2\frac{(k-1)}{n}\frac{\left\{M_{x_2}f_{x_2}(M_{x_2})\right\}^{-2}}{4},\ B= f_3\frac{\left\{M_zf_z(M_z)\right\}^{-2}}{4}+W_2\frac{(k-1)}{n}\frac{\left\{M_{z_2}f_{z_2}(M_{z_2})\right\}^{-2}}{4}$$

$$C= f_3\frac{\rho_{xy}\left\{M_xM_yf_x(M_x)f_y(M_y)\right\}^{-1}}{4}+W_2\frac{(k-1)}{n}\frac{\rho_{xy(2)}\left\{M_{x_2}M_{y_2}f_{x_2}(M_{x_2})f_{y_2}(M_{y_2})\right\}^{-1}}{4},$$

$$D= f_3\frac{\rho_{yz}\left\{M_yM_zf_y(M_y)f_z(M_z)\right\}^{-1}}{4}+W_2\frac{(k-1)}{n}\frac{\rho_{yz(2)}\left\{M_{y_2}M_{z_2}f_{y_2}(M_{y_2})f_{z_2}(M_{z_2})\right\}^{-1}}{4},$$

$$E = f_3\frac{\rho_{xz}\left\{M_xM_zf_x(M_x)f_z(M_z)\right\}^{-1}}{4}+W_2\frac{(k-1)}{n}\frac{\rho_{xz(2)}\left\{M_{x_2}M_{z_2}f_{x_2}(M_{x_2})f_{z_2}(M_{z_2})\right\}^{-1}}{4},$$

$$A'=f_1\frac{\left\{M_xf_x(M_x)\right\}^{-2}}{4}+W_2\frac{(k-1)}{n}\frac{\left\{M_{x_2}f_{x_2}(M_{x_2})\right\}^{-2}}{4},\ B'= f_1\frac{\left\{M_zf_z(M_z)\right\}^{-2}}{4}+W_2\frac{(k-1)}{n}\frac{\left\{M_{z_2}f_{z_2}(M_{z_2})\right\}^{-2}}{4},$$

$$C'=f_1\frac{\rho_{xy}\left\{M_xM_yf_x(M_x)f_y(M_y)\right\}^{-1}}{4}+W_2\frac{(k-1)}{n}\frac{\rho_{xy(2)}\left\{M_{x_2}M_{y_2}f_{x_2}(M_{x_2})f_{y_2}(M_{y_2})\right\}^{-1}}{4},$$

$$D'= f_1\frac{\rho_{yz}\left\{M_yM_zf_y(M_y)f_z(M_z)\right\}^{-1}}{4}+W_2\frac{(k-1)}{n}\frac{\rho_{yz(2)}\left\{M_{y_2}M_{z_2}f_{y_2}(M_{y_2})f_{z_2}(M_{z_2})\right\}^{-1}}{4}$$

$$\text{and } E' =f_1\frac{\rho_{xz}\left\{M_xM_zf_x(M_x)f_z(M_z)\right\}^{-1}}{4}+W_2\frac{(k-1)}{n}\frac{\rho_{xz(2)}\left\{M_{x_2}M_{z_2}f_{x_2}(M_{x_2})f_{z_2}(M_{z_2})\right\}^{-1}}{4}$$

Remark 3.1.

The bias and mean square errors of the various estimators (indicated in section 2) belonging to the classes of estimators $T_i\left(i = 1, 2, \ldots, 4\right)$ can be easily obtained by substituting the suitable values of the derivatives in equations (24) - (31) as suggested in population mean estimation technique by Singh *et al.* (2007).

4. Minimum M. S. E. of the Classes of Estimators $T_i\left(i = 1, 2, \ldots, 4\right)$

It is obvious from the equations (28) - (31) and remark 3.1 that the mean square errors of the proposed classes of estimators $T_i\left(i = 1, 2, \ldots, 4\right)$ depend on the different values of the derivatives $d_2, d_4, c_2, c_3, p_2, p_4, q_2$ and q_3. Therefore, we desire to minimize the mean square errors of the classes of estimators T_i. We differentiate the equations (28) – (31) with respect to $d_2, d_4, c_2, c_3, p_2, p_4, q_2$ and q_3 and equate the results to zero. Thus, we have obtained the optimum values of $d_2, d_4, c_2, c_3, p_2, p_4, q_2$ and q_3 for minimizing the M. S. E. of the classes of estimators $T_i\left(i = 1, 2, \ldots, 4\right)$ as

$$\left.\begin{aligned} &d_2= -\frac{M_yC}{M_xA},\ d_4= -\rho_{yz}\frac{f_z(M_z)}{f_y(M_y)},\ c_2=\frac{M_y\left(ED - BC\right)}{M_x\left(AB - E^2\right)},\ c_3=\frac{M_y\left(EC- AD\right)}{M_z\left(AB- E^2\right)},\\ &p_2= -\rho_{xy}\frac{f_x(M_x)}{f_y(M_y)},\ p_4= d_4,\ q_2=\frac{M_y\left(E''D - BC''\right)}{M_x\left\{A''B - E''^2\right\}}\text{ and } q_3=\frac{M_y\left(E''C''- A''D\right)}{M_z\left\{A''B - E''^2\right\}}\end{aligned}\right\} \quad (32)$$

where

$A''=\frac{f_3}{4}\left\{M_x f_x\left(M_x\right)\right\}^{-2}$, $C''=\frac{f_3}{4}\rho_{xy}\left\{M_x M_y f_x(M_x) f_y(M_y)\right\}^{-1}$ and $E''=\frac{f_3}{4}\rho_{xz}\left\{M_x M_z f_x(M_x) f_z(M_z)\right\}^{-1}$.

Substituting these optimum values of the derivatives in equations (28) - (31), we have minimum M. S. E.s of the classes of estimators $T_i\left(i=1, 2, \ldots, 4\right)$ as

$$\text{Min. } M\left(T_1\right)=M_y^2\left\{f_1\frac{\left\{M_y f_y(M_y)\right\}^{-2}}{4}+W_2\frac{\left(k\text{-}1\right)}{n}\frac{\left\{M_{y_2} f_{y_2}(M_{y_2})\right\}^{-2}}{4}\right\}-M_y^2 f_2\frac{\rho_{yz}^2}{4}\left\{M_y f_y(M_y)\right\}^{-2} \tag{33}$$

$$-M_y^2\frac{C^2}{A},$$

$$\text{Min. } M\left(T_2\right)=M_y^2\left\{f_1\frac{\left\{M_y f_y(M_y)\right\}^{-2}}{4}+W_2\frac{\left(k\text{-}1\right)}{n}\frac{\left\{M_{y_2} f_{y_2}(M_{y_2})\right\}^{-2}}{4}\right\}-M_y^2\frac{AD^2+BC^2-2EDC}{AB-E^2}, \tag{34}$$

$$\text{Min. } M\left(T_3\right)=M_y^2\left\{f_1\frac{\left\{M_y f_y(M_y)\right\}^{-2}}{4}+W_2\frac{\left(k\text{-}1\right)}{n}\frac{\left\{M_{y_2} f_{y_2}(M_{y_2})\right\}^{-2}}{4}\right\}-f_2\frac{M_y^2}{4}\left[\rho_{yz}\left\{M_y f_y(M_y)\right\}^{-1}\right]^2$$

$$-f_3\frac{M_y^2}{4}\left[\rho_{xy}\left\{M_y f_y(M_y)\right\}^{-1}\right]^2 \tag{35}$$

and

$$\text{Min. } M(T_4)=M_y^2\left\{f_1\frac{\left\{M_y f_y(M_y)\right\}^{-2}}{4}+W_2\frac{\left(k\text{-}1\right)}{n}\frac{\left\{M_{y_2} f_{y_2}(M_{y_2})\right\}^{-2}}{4}\right\}-M_y^2\frac{A''D^2+BC''^2-2DE''C''}{\left\{A''B-E''^2\right\}}. \tag{36}$$

Remark 4.1: It is to be noted from the optimality conditions in equation (32) that optimum values of derivatives of the proposed classes of estimators $T_i\left(i=1, 2, \ldots, 4\right)$ depend on unknown population parameters such as M_x, M_y, M_z, M_{x_2}, M_{y_2}, M_{z_2}, $f_x(M_x), f_y(M_y)$, ρ_{xz}, $\rho_{xy(2)}, \rho_{yz(2)}$ and $\rho_{xz(2)}$. Thus, to use such estimators one has to use guessed or estimated values of these parameters. Guessed values of these population parameters can be obtained either from past data or experience gathered over time; see Murthy (1967) and Tracy et al. (1996). If such guessed values are not available then it is advisable to use sample data to estimate these parameters as suggested by Silverman (1986) and Singh *et al.* (2001). In case, non-response situations occur in the sample data, it is advised to utilize the sub-sampling of the non-responding group technique to estimate these parameters as suggested in this paper. It could be seen that the mean square errors of the proposed classes of estimators remains same up to the first order of approximations, even if population parameters are replaced by their respective sample estimates.

5. Efficiency Comparisons of the Proposed Classes of Estimators $T_i\left(i=1, 2, \ldots, 4\right)$

It is important to investigate the situations under which our proposed classes of estimators $T_i\left(i=1, 2, \ldots, 4\right)$ are preferable over the estimators considered under the similar situations such as

$\hat{M}_y^*$ and t_i^* and t_i^{**} (i = 1, 2). Proceeding as sections 3 and 4, the variance V(.)/minimum V(.)/ M. S. E.s of the estimators t_i to the first order of approximations are obtained as

$$V\left(M_y^*\right)= M_y^2\left\{f_1\frac{\left\{M_y f_y(M_y)\right\}^{-2}}{4} + W_2\frac{(k-1)}{n}\frac{\left\{M_{y_2} f_{y_2}(M_{y_2})\right\}^{-2}}{4}\right\} \quad (37)$$

$$M(t_1^*) = M_y^2\left[f_1\frac{\left(M_y f_y(M_y)\right)^{-2}}{4} + \frac{W_2(k-1)}{n}\frac{\left(M_{y_2} f_{y_2}(M_{y_2})\right)^{-2}}{4} + A - 2C\right] \quad (38)$$

$$\text{Min. } V(t_2^*) = M_y^2\left[f_1\frac{\left(M_y f_y(M_y)\right)^{-2}}{4} + \frac{W_2(k-1)}{n}\frac{\left(M_{y_2} f_{y_2}(M_{y_2})\right)^{-2}}{4} - \frac{C^2}{A}\right] \quad (39)$$

$$M(t_1^{**}) = M_y^2\left[\begin{array}{l} f_1\frac{\left(M_y f_y(M_y)\right)^{-2}}{4} + \frac{W_2(k-1)}{n}\frac{\left(M_{y_2} f_{y_2}(M_{y_2})\right)^{-2}}{4} + f_3\frac{\left(M_x f_x(M_x)\right)^{-2}}{4} \\ -2f_3\frac{\rho_{xy}\left\{M_x M_y f_x(M_x) f_y(M_y)\right\}^{-1}}{4} \end{array}\right] \quad (40)$$

$$\text{Min. } V(t_2^{**}) = M_y^2\left[\frac{f_1\left(M_y f_y(M_y)\right)^{-2}}{4} + \frac{W_2(k-1)}{n}\frac{\left(M_{y_2} f_{y_2}(M_{y_2})\right)^{-2}}{4} - f_3\frac{\rho_{xy}^2}{4}\left(M_y f_y(M_y)\right)^{-2}\right] \quad (41)$$

5.1. Efficiency Comparisons of the Classes of Estimators T_1 and T_2

When non-response situations is observed on the study variable y as well as on the auxiliary variable x in the second phase sample of size n, we compare the efficiencies of our proposed classes of estimators $T_i (i = 1, 2)$ under their respective optimality conditions with the estimators $\hat{M}_y^*$, $t_i^* (i = 1, 2)$ and present them below.

(a) Efficiency Comparisons of the class of estimators T_1 :

It could be concluded from equations (33) and (37) - (39) that

(i) T_1 is more efficient than $\hat{M}_y^*$, provided $V\left(\hat{M}_y^*\right) - \text{Min. } M\left(T_1\right) > 0$

which is possible when

$$M_y^2\left\{f_2\frac{\rho_{yz}^2}{4}\left\{M_y f_y(M_y)\right\}^{-2} + \frac{C^2}{A}\right\} \geq 0. \quad (42)$$

It can be observed from equation (42) that the class of estimators T_1 is always preferable over $\hat{M}_y^*$, as $\left(f_1, f_2, f_3 > 0\right)$ and $A = f_3\frac{\left\{M_x f_x(M_x)\right\}^{-2}}{4} + W_2\frac{(k-1)}{n}\frac{\left\{M_{x_2} f_{x_2}(M_{x_2})\right\}^{-2}}{4} > 0$ always.

(ii) T_1 is more precise than t_1^* provided $M\left(t_1^*\right) - \text{Min. } M\left(T_1\right) > 0.$

which is probable if

$$\frac{C^2}{A} + A - 2C \geq 0 \Rightarrow \frac{C^2 + A^2 - 2AC}{A} \geq 0 \quad \Rightarrow \frac{(C-A)^2}{A} \geq 0. \quad (43)$$

It may be noted from the above equation that T_1 is always more efficient than the estimator t_1^* as $A > 0$ always.
Similarly,
(iii) T_1 is always preferable over t_2^* as

$$M_y^2 f_2 \frac{\rho_{yz}^2}{4}\left\{M_y f_y(M_y)\right\}^{-2} \geq 0 \text{ always.} \tag{44}$$

(b) Efficiency Comparisons of T_2 :
Proceeding as above, it can be observed from equations that (34), and (37) that
(i). T_2 is more efficient than the estimators $\hat{M}_y^*$ provided

$$V\left(\hat{M}_y^*\right) - \text{Min. } M\left(T_2\right) > 0$$

$$\Rightarrow \quad \frac{AD^2 + BC^2 - 2EDC}{AB - E^2} \geq 0 \tag{45}$$

which is possible when $AD^2 + BC^2 - 2EDC \geq 0$ and $AB - E^2 > 0$.
Now
$AD^2 + BC^2 - 2EDC \geq 0$

$$\Rightarrow \frac{f_3}{4}\left\{\left(M_x f_x(M_x)\right)^{-2} D^2 + \left(M_z f_z(M_z)\right)^{-2} C^2 - 2DC\rho_{xz}\left(M_x M_z f_x(M_x) f_z(M_z)\right)^{-1}\right\}$$
$$+W_2 \frac{(k-1)}{4n}\left\{\left(M_{x_2} f_{x_2}(M_{x_2})\right)^{-2} D^2 + \left(M_{z_2} f_{z_2}(M_{z_2})\right)^{-2} C^2 - 2DC\rho_{x_2 z_2}\left(M_{x_2} M_{z_2} f_{x_2}(M_{x_2}) f_{z_2}(M_{z_2})\right)^{-1}\right\} \geq 0.$$

and
$AB - E^2 > 0$

$$\Rightarrow \frac{f_3^2}{16}\left\{\left(M_x M_z f_x(M_x) f_z(M_z)\right)^{-2}\left(1 - \rho_{xz}^2\right)\right\} + \left\{W_2 \frac{(k-1)}{4n}\right\}^2 \left\{\left(M_{x_2} M_{z_2} f_{x_2}(M_{x_2}) f_{z_2}(M_{z_2})\right)^{-2}\left(1 - \rho_{xz(2)}^2\right)\right\}$$
$$+ f_3 W_2 \frac{(k-1)}{16n}\left\{\begin{array}{l}\left(M_x M_{z_2} f_x(M_x) f_{z_2}(M_{z_2})\right)^{-2} + \left(M_{x_2} M_z f_{x_2}(M_{x_2}) f_z(M_z)\right)^{-2} \\ -2\rho_{xz}\rho_{xz(2)}\left(M_x M_{z_2} f_x(M_x) f_{z_2}(M_{z_2}) M_{x_2} M_z f_{x_2}(M_{x_2}) f_z(M_z)\right)\end{array}\right\} > 0$$

The conditions stated in equation (45) occur always provided $\left(-1<\rho_{xz}<1, -1<\rho_{xz(2)}<1\right)$. Similarly,
(ii) From equations (34) and (39), it can be found that T_2 is more precise than t_2^* if

$$\frac{AD^2 + BC^2 - 2EDC}{AB - E^2} - \frac{C^2}{A} > 0 \tag{46}$$

which is possible only when
$A^2D^2 + E^2C^2 - 2ADEC > 0$ and $AB - E^2 > 0$
i.e. when $AD \neq EC$ and $\left(-1<\rho_{xz}, \rho_{xz(2)}<1\right)$ as $\left(\rho_{xz}, \rho_{xz(2)} = 1\right) \Rightarrow AB - E^2 = 0.$
(iii) From equations (34) and (38), it is observed that T_2 is preferable than t_1^* when

$$A - 2C + \frac{AD^2 + BC^2 - 2EDC}{AB - E^2} > 0. \tag{47}$$

Now, following the efficiency comparisons mentioned in equations (43) and (46), it may be noticed that

$$\frac{AD^2 + BC^2 - 2EDC}{AB - E^2} > \frac{C^2}{A} \Rightarrow \frac{AD^2 + BC^2 - 2EDC}{AB - E^2} + A - 2C > 0.$$

Therefore the class of estimators T_2 is more efficient than t_1^* if the conditions $AD \neq EC$ and $\left(-1<\rho_{xz}, \rho_{xz(2)}<1\right)$ are met.

5.2. Efficiency Comparisons of the Classes of Estimators T_3 and T_4

We wish to compare the efficiencies of the classes of estimators T_3 and T_4 under their respective optimality conditions with the estimators $\hat{M}_y^*$ and $t_i^{**}(i = 1, 2)$ when there is non-response only on the study variable y but the complete information on the auxiliary variable x is available in the second phase sample of size n.

(a) Efficiency Comparisons of T_3:

Proceeding as above for the comparisons of efficiencies of the class of estimators T_3 with the estimators $\hat{M}_y^*$ and $t_i^{**}(i = 1, 2)$ from equations (35), (37), (40) and (41), we observe that T_3 is more efficient than them as $V\left(\hat{M}_y^*\right) - \text{Min. } M\left(T_3\right)$, $M\left(t_1^{**}\right) - \text{Min. } M\left(T_3\right)$ and $\text{Min. } V\left(t_2^{**}\right) - \text{Min. } M\left(T_3\right)$ are always positive.

(b) Efficiency Comparisons of T_4:

(i) Comparisons of efficiencies of the class of estimators T_4 with the estimator $\hat{M}_y^*$ from equations (36) and (37) revels the fact that the class of estimators T_4 is preferable over the estimators $\hat{M}_y^*$ provided

$V\left(\hat{M}_y^*\right) - \text{Min. } M\left(T_4\right) > 0$

which is possible only when

$$\frac{A''D^2 + BC''^2 - 2DE''C''}{\left\{A''B - E''^2\right\}} \geq 0. \qquad (48)$$

Following the efficiency comparison technique discussed in section 5.1. (b), it can be easily verified that the condition stated in equation (48) occurs when $\left(-1 < \rho_{xz} < 1\right)$.

(ii) Similarly, a comparisons of efficiencies of the class of estimators T_4 with the estimator t_2^{**} from equations (36) and (41) indicates that the class of estimators T_4 is more efficient than the estimators t_2^{**} provided

$$\frac{A''D^2 + BC''^2 - 2DE''C''}{A''B - E''^2} - \frac{f_3\rho_{xy}^2\left\{M_y f_y(M_y)\right\}^{-2}}{4} > 0 \qquad (49)$$

$$\Rightarrow \frac{A''D^2 + BC''^2 - 2DE''C''}{A''B - E''^2} - \frac{C''^2}{A''} > 0 \Rightarrow \frac{\left(A''D - E''C''\right)^2}{A''B - E''^2} > 0$$

It can be found that the above condition is satisfied when $\rho_{yz} \neq \rho_{xy}\rho_{xz}$ and $-1 < \rho_{xz} < 1$.

(iii) Proceeding as the efficiency comparisons of the classes of estimators T_2 (with t_1^{**}) and T_4 (with t_2^{**}) discussed above, it can be easily established the class of estimators T_4 is more efficient than the estimator t_1^{**} provided $\rho_{yz} \neq \rho_{xy}\rho_{xz}$ and $-1 < \rho_{xz} < 1$.

6. Numerical Illustrations

We have chosen three natural population data sets and one artificially generated population data set to illustrate the efficacious performances of our proposed classes of estimators. The source of the populations, the nature of the variables y, x, z and the values of the various parameters are given as follows.

Natural population data sets

Population I- Source: Statistical Abstract of the United States, 2012 (Table No. 233)

The present data belongs to the state wise educational attainment of United States in the year 2012. Advanced degree or more in the year 2009 is taken as study variable y while Bachelor's degree or more in the years 2007 and 2006 are taken as auxiliary variables x and z respectively. The first 10 states have been considered as non-responding part of the population for the variables where non-responses occur.

Population II- Source: Statistical Abstract of the United States, 2012 (Table No. 629)

The state wise total unemployment of civilian labour force of United States in the year 2012 has been taken under study. Percent of total unemployed of the year 2010, 2009 and 2008 are considered as the variables y, x and z respectively. The first 10 states have been considered as non-responding group of the population for the variables where non-responses occur.

Population III- Source: Cochran (1977), pp.-34

This data set indicates the weakly expenditure (y), the number of person (x) and the weakly family income (z) of 33 low income families. The first 7 families are considered as non-responding group of the population for the variables where non-response situations found.

Table1: Parametric values of different populations.

Population	M_y	M_x	M_z	M_{y_2}	M_{x_2}	M_{z_2}	$f_y(M_y)$	$f_x(M_x)$	$f_z(M_z)$
Population-I	9.3	25.8	25.6	10.00	26.05	26.95	0.0727	0.0728	0.1080
N= 51, W_2=0.1961,	$f_{y_2}(M_{y_2})$	$f_{x_2}(M_{x_2})$	$f_{z_2}(M_{z_2})$	ρ_{xy}	ρ_{yz}	ρ_{xz}	$\rho_{xy(2)}$	$\rho_{yz(2)}$	$\rho_{xz(2)}$
n'=25, n=12	0.0639	0.0476	0.0524	0.9100	0.9008	0.9956	0.9573	0.9508	0.9958
Population-II	M_y	M_x	M_z	M_{y_2}	M_{x_2}	M_{z_2}	$f_y(M_y)$	$f_x(M_x)$	$f_z(M_z)$
N= 51,	8.7	8.2	5.3	9.3	8.95	5.75	0.1918	0.2024	0.3260
W_2=0.1961,	$f_{y_2}(M_{y_2})$	$f_{x_2}(M_{x_2})$	$f_{z_2}(M_{z_2})$	ρ_{xy}	ρ_{yz}	ρ_{xz}	$\rho_{xy(2)}$	$\rho_{yz(2)}$	$\rho_{xz(2)}$
n'=25, n=12	0.2831	0.3358	0.5272	0.9516	0.8652	0.9163	0.9586	0.6397	0.6090
Population-III	M_y	M_x	M_z	M_{y_2}	M_{x_2}	M_{z_2}	$f_y(M_y)$	$f_x(M_x)$	$f_z(M_z)$
N= 33,	24.2	4	69	24.2	3	65	0.0380	0.2610	0.0361
W_2=0.2121,	$f_{y_2}(M_{y_2})$	$f_{x_2}(M_{x_2})$	$f_{z_2}(M_{z_2})$	ρ_{xy}	ρ_{yz}	ρ_{xz}	$\rho_{xy(2)}$	$\rho_{yz(2)}$	$\rho_{xz(2)}$
n'=20, n=10	0.0494	0.2184	0.0241	0.4327	0.2522	-0.066	0.5071	-0.1450	0.2434

For completing the data sets of above populations we have taken

$$f_y(M_y)=\frac{1}{\sqrt{2\pi}\sigma_y}e^{-\frac{1}{2}\left(\frac{M_y-\mu_y}{\sigma_y}\right)^2},\ f_x(M_x)=\frac{1}{\sqrt{2\pi}\sigma_x}e^{-\frac{1}{2}\left(\frac{M_x-\mu_x}{\sigma_x}\right)^2},\ f_z(M_z)=\frac{1}{\sqrt{2\pi}\sigma_z}e^{-\frac{1}{2}\left(\frac{M_z-\mu_z}{\sigma_z}\right)^2},$$

$$f_{y_2}(M_{y_2})=\frac{1}{\sqrt{2\pi}\sigma_{y_2}}e^{-\frac{1}{2}\left(\frac{M_{y_2}-\mu_{y_2}}{\sigma_{y_2}}\right)^2},\qquad f_{x_2}(M_{x_2})=\frac{1}{\sqrt{2\pi}\sigma_{x_2}}e^{-\frac{1}{2}\left(\frac{M_{x_2}-\mu_{x_2}}{\sigma_{x_2}}\right)^2}$$

and $f_{z_2}(M_{z_2})=\frac{1}{\sqrt{2\pi}\sigma_{z_2}}e^{-\frac{1}{2}\left(\frac{M_{z_2}-\mu_{z_2}}{\sigma_{z_2}}\right)^2}$ {see for instance Singh and Priyanka (2008)}

where

μ_y, μ_x, μ_z : Population means of the variables y, x and z based on the whole population U.

$\sigma_y, \sigma_x, \sigma_z$:Population standard deviations of the respective variables based on the whole population.

$\mu_{y_2}, \mu_{x_2}, \mu_{z_2}$: Population means of the variables y, x and z based on the non-responding part of the population i. e. U_2.

$\sigma_{y_2}, \sigma_{x_2}, \sigma_{z_2}$: Population standard deviations of the respective variables based on the non-responding part of the population.

Artificially Generated Population

We have generated three sets of independent random numbers of size N (N = 100) namely x'_k, y'_k and z'_k (k =1, 2, 3, . . ., N) from a standard normal distribution with the help of R-software. Further, motivated by the artificial population generation techniques adopted by Singh and Deo (2003) and Singh *et al.* (2001), we have generated the following transformed variables of the population U with the values of $\rho_{xy}=0.8$, $\rho_{xz}=0.6$, $\sigma_y^2=100$, $\sigma_x^2=225$, $\mu_y=40$, $\mu_x=50$, $\sigma_z^2=25$ and

$\mu_z = 30$ as

$$y_k = \mu_y + \sigma_y \left[\rho_{xy} x'_k + \left(\sqrt{1-\rho_{xy}^2} \right) y'_k \right],\ x_k = \mu_x + \sigma_y x'_k \text{ and } z_k = \mu_z + \sigma_z \left[\rho_{xz} x'_k + \left(\sqrt{1-\rho_{xz}^2} \right) z'_k \right] \quad (50)$$

We assume that randomly r % of the whole population (i. e. U) shows non-responses which constitute the data set of the non - responding group (i. e. U_2) for the variables where non-responses occur.

To have a tangible idea about the performance of the proposed classes of estimators $T_i (i = 1, 2, \ldots, 4)$, we have computed the percent relative efficiencies (PREs) of the estimators T_i and the estimators t_i^* and $t_i^{**} (i = 1, 2)$ under their respective optimality conditions with respect to the sample median estimator $\hat{M}_y^*$. The findings are displayed in tables 2 and 3 where we have designated the percent relative efficiencies (PREs) of an estimator of t with respect to $\hat{M}_y^*$ as

$$\text{PRE} = \left(V(\hat{M}_y^*) \big/ M_v(t) \right) \times 100 \quad (51)$$

and $M_v(t)$ denotes the M. S. E./ Minimum M. S. E. /Minimum variance of an estimator t.

Table 2: PREs of the different estimators with respect to $\hat{M}_y^*$ for natural population data sets.

Population I	Non-Response on both y and x				Non-Response on only y			
	t_1^*	t_2^*	T_1	T_2	t_1^{**}	t_2^{**}	T_3	T_4
k = 2	178.0707	268.4340	585.8057	278.8765	138.4715	177.8557	277.4483	273.9718
k = 3	196.0538	308.4413	628.3841	322.7996	129.4055	155.7707	209.6889	316.4425
k = 4	210.7819	345.9819	669.3903	362.5431	123.7975	143.4464	179.3780	354.4708
k = 5	223.0654	380.7883	706.1602	398.7206	119.9860	135.5832	162.1922	388.7155
Population II	t_1^*	t_2^*	T_1	T_2	t_1^{**}	t_2^{**}	T_3	T_4
k = 2	279.0659	279.8741	713.5251	279.8991	225.4662	226.3891	445.3089	233.8416
k = 3	298.2974	298.7254	734.4845	298.7257	203.6422	204.3302	343.8811	222.0368
k = 4	316.6866	316.8653	754.0629	316.8981	188.2855	188.8271	288.5100	214.6977
k = 5	334.2877	334.3290	772.3270	334.4421	176.8923	177.3353	253.6296	209.4665
Population III	t_1^*	t_2^*	T_1	T_2	t_1^{**}	t_2^{**}	T_3	T_4
k = 2	*	116.5442	118.6496	117.0422	*	112.8446	114.8172	114.4438
k = 3	*	118.2245	120.0999	118.2588	*	110.9565	112.6068	111.3065
k = 4	*	119.7417	121.4377	120.2838	*	109.5523	110.9703	109.5919
k = 5	*	121.0396	122.5890	122.3084	*	108.4672	109.7098	108.4712

Note: “*” indicates no gain, i.e., PRE is less than 100%.

Table 3: PREs of the different estimators with respect to $\hat{M}_y^*$ for artificially generated population taking $n' = 30$ and $n = 15$.

	Non-Response on both y and x				Non-Response on only y			
r =5%	t_1^*	t_2^*	T_1	T_2	t_1^{**}	t_2^{**}	T_3	T_4
k = 2	135.9663	161.2845	189.6015	161.2968	143.3703	161.0258	189.2441	161.3929
k = 3	128.5964	160.4652	187.8670	160.4767	142.2352	159.2378	186.1868	160.5828
k = 4	122.2115	160.1780	186.8981	160.1859	141.1581	157.5515	183.3319	160.1765
k = 5	116.6264	160.2526	186.4474	160.2564	140.1345	155.9586	180.6602	160.0142
r = 10%	t_1^*	t_2^*	T_1	T_2	t_1^{**}	t_2^{**}	T_3	T_4
k = 2	162.2977	182.1849	212.3261	182.9148	134.1463	146.8205	165.7868	155.0777
k = 3	177.6132	199.0628	229.3572	201.8821	127.6748	137.2795	151.0373	158.7334
k = 4	190.9762	213.9396	244.1136	219.7776	123.2655	130.9687	141.6902	165.4765
k = 5	202.7377	227.1370	257.0036	236.6420	120.0681	126.4852	135.2369	173.1624
r = 15%	t_1^*	t_2^*	T_1	T_2	t_1^{**}	t_2^{**}	T_3	T_4
k = 2	143.0242	168.2173	197.1078	168.4216	139.7904	155.4252	179.7709	156.2917
k = 3	141.7262	173.2925	201.5592	173.8196	135.9379	149.5227	170.1062	151.9618
k = 4	140.6198	178.0961	205.7831	178.9933	132.7656	144.7563	162.5303	148.7460
k = 5	139.6654	182.6183	209.7519	183.9033	130.1079	140.8269	156.4321	146.2010
r = 20%	t_1^*	t_2^*	T_1	T_2	t_1^{**}	t_2^{**}	T_3	T_4
k = 2	134.8370	162.1525	186.0803	162.1857	134.9635	148.0497	167.7436	153.4308
k = 3	128.5662	162.2857	182.5433	162.4713	128.7646	138.8626	153.4321	150.4753
k = 4	124.1886	162.6687	180.2970	163.0722	124.4328	132.6248	144.1128	149.0175
k = 5	120.9593	163.1079	178.7393	163.7516	121.2349	128.1124	137.5615	148.1431

7. Conclusions

The following conclusions can be read-out from the present study.

From efficiency comparisons in the section 5, it is observed that:

Suggested classes of estimators $T_i\,(i = 1, 2, \ldots, 4)$ are always more efficient than the sample median estimator $\hat{M}_y^*$ under their respective optimality conditions and the classes of estimators T_1 and T_3 are always preferable over the estimators t_i^* and t_i^{**} (i = 1, 2) respectively under similar non – response situations.

From Table 2, it is vindicated that:

For different values of the correlation coefficients (i. e. ρ_{yz}, ρ_{xy}, ρ_{xz}, $\rho_{yz(2)}$, $\rho_{xy(2)}$ and $\rho_{xz(2)}$) and under the similar non - response situations, proposed classes of estimators $T_i\,(i = 1, 2, \ldots, 4)$ are more efficient than the estimators $\hat{M}_y^*$, t_i^* and t_i^{**} (i = 1, 2). It is also noted that for high positive values these correlation coefficients, the classes of estimators T_i yield substantial gain in efficiency over the estimator $\hat{M}_y^*$ (especially visible from the populations I and II).

From Table 3, it is clear that:

For different percentages of non - response rate r and different values of the sub - sampling fraction $\left(\frac{1}{k}\right)$ the classes of estimators T_i (i= 1, 2, . . ., 4) are preferable over the estimators $\hat{M}_y^*$, t_i^* and t_i^{**} (i = 1, 2).

Hence, the proposals of the classes of estimators in the present study are more justifiable in compare with the previous work of similar nature as they unify several desirable results including effectively handling of the various realistic situations of non - responses. Therefore, they may be recommended to the survey statisticians and practitioners for their applications in real life problems.

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