

## On Some Properties and Applications of Intervened Gegenbauer Distribution

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Received: 18<sup>th</sup> August 2017 / Revised: 04<sup>th</sup> December 2018 / Accepted: 20<sup>th</sup> May 2019  
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### ABSTRACT

*In this paper, an intervened version of the Gegenbauer distribution is considered and investigated some of its statistical properties. The parameters of the distribution are estimated by the method of maximum likelihood and illustrated using real life data sets. The likelihood ratio test procedure is applied for examining the significance of the intervention parameters and a simulation study is carried out for assessing the performances of the maximum likelihood estimators.*

**Keywords** : Gegenbauer distribution, Gegenbauer polynomials, intervened negative binomial distribution, maximum likelihood estimation, probability generating functions.

AMS (2000) Mathematics Subject Classification : 60E05, 60E10.

### 1. Introduction

Intervened type distributions have received much attention in the literature due to their extensive practical utility. For example these types of distributions have found utility in the study of effectiveness of different types of treatments in connection with various diseases. Also they have been utilized in certain studies on advertisement effectiveness and in quality control for controlling the defective items. (Bartolucci et al, 2001) developed an intervened geometric distribution (IGD) as a modification to zero-truncated geometric distribution in connection with a cardiovascular study. (Kumar and Sreeja, 2014, 2016)

studied some modified versions of the IGD. (Kumar and Sreeja, 2012) have proposed the intervened negative binomial distribution (INBD) as a generalization of the IGD. The INBD is the distribution of the sum of a zero truncated negative binomial random variable and an independent negative binomial random variable. That is, a random variable  $U$  is said to follow the INBD if its probability mass function (pmf)  $f_u$  has the following form, for  $u = 1, 2, 3, \dots$

$$f_u = \frac{(1-\rho\theta)^r \theta^u}{[(1-\theta)^{-r} - 1][\Gamma(r)]^2} \sum_{j=0}^{u-1} \frac{\Gamma(u-j+r)\Gamma(j+r)\rho^j}{\Gamma(u-j+1)\Gamma(j+1)}$$

in which  $r > 0$ ,  $\rho > 0$ ,  $0 < \theta < 1$  and  $0 < \rho\theta < 1$ . When  $r = 1$  the INBD reduces to the IGD. The mean and variance of the INBD is

$$\begin{aligned} \text{mean} &= r\delta_2 + r\delta_0 \delta_1, \\ \text{variance} &= r\delta_2(1 + \delta_2) + r\delta_0 \delta_1(1 + \delta_1) + \delta_0(1 - \delta_0)(r\delta_1)^2, \end{aligned}$$

in which  $\delta_0 = [1 - (1 - \theta)^{-1}]^{-1}$ ,  $\delta_1 = \theta(1 - \theta)^{-1}$  and  $\delta_2 = \rho\theta(1 - \rho\theta)^{-1}$ . The INBD is over-dispersed or under-dispersed depending on the values of the parameters and in practical situations INBD is considered as more suitable than zero truncated negative binomial distribution (ZTNBD). For example, in epidemiological studies the INBD provides information on the effectiveness of preventive actions taken by health agencies where ZTNBD fails. (Kumar and Sreeja, 2017) studied about a modified version of the INBD. (Plunket and Jain, 1975) derived the Gegenbauer distribution with probability generating function (pgf) of the form

$$P(s) = (1 - \theta_1 - \theta_2)^r (1 - \theta_1 s - \theta_2 s^2)^{-r}, \quad (1.1)$$

where  $r > 0$ ,  $\theta_1 > 0$  and  $\theta_2 > 0$  such that  $\theta_1 + \theta_2 \leq 1$ . When  $\theta_2 = 0$  the Gegenbauer distribution reduces to the negative binomial distribution.

The INBD is suitable for single intervention situations. But there are many practical situations where intervention occurs in multiple form. So in order to model such situations a more general class of the INBD is required. So through this paper we propose a model as a generalization of the INBD which is suitable for tackling multiple intervention situations, named as “the intervened Gegenbauer distribution” or in short “the IGbD” The paper is organized in such a way that in section 2 we develop a model leading to the IGbD and derive expressions for the pmf, moments, mean, variance and recurrence relations of probabilities. In section 3 we discuss the estimation of the parameters of the IGbD by the method of maximum likelihood. A simulation study is carried out in section 4 for

examining the performance of the estimators of the parameters of the IGbD and to know the effectiveness of intervention parameters. A discussion part is included in section 5.

We need the following in the sequel. For any real valued function  $A(i; r)$ ; we have

$$\sum_{i=0}^{\infty} \sum_{r=0}^{\infty} A(i, r) = \sum_{i=0}^{\infty} \sum_{r=0}^i A(i - r, r) \tag{1.2}$$

and for any  $\lambda > 0$ ,  $a > 0$  and  $b > 0$  such that  $a + b < 1$ ; the Gegenbauer polynomials  $G_n^\lambda(a, b)$  defined through the generating function

$$G(s) = (1 - as - bs^2)^{-\lambda} = \sum_{n=0}^{\infty} G_n^\lambda(a, b) s^n ,$$

in which

$$G_n^\lambda(a, b) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\Gamma(\lambda + n - j)}{\Gamma(j + 1)\Gamma(n - 2j + 1)\Gamma(\lambda)} a^{n-2j} b^j ,$$

where  $[k]$  denotes the integer part of  $k$ . For details of  $G_n^\lambda(a, b)$  see Plunket and Jain(1975).

## 2 . Intervened Gegenbauer Distribution

Let  $Z$  be a random variable having zero truncated Gegenbauer distribution (ZTGbD) with parameters  $r, \theta_1$  and  $\theta_2$ . Then the pgf of  $Z$  is

$$P_Z(s) = \frac{(1 - \theta_1 - \theta_2)^r}{1 - (1 - \theta_1 - \theta_2)^r} [(1 - \theta_1 s - \theta_2 s^2)^{-r} - 1] \tag{2.1}$$

in which  $r > 0$  and  $\theta_i > 0$  for  $i = 1, 2$  such that  $\theta_1 + \theta_2 \leq 1$ .

Let  $Y$  be a random variable following Gegenbauer distribution in which, due to some intervention, the parameters  $\theta_1$  changes to  $\rho_1\theta_1$  and  $\theta_2$  changes to  $\rho_2\theta_2$  with  $\rho_1 > 0, \rho_2 > 0$  such that  $\rho_1\theta_1 + \rho_2\theta_2 \leq 1$  and the parameters  $\rho_1$  and  $\rho_2$  are called the intervention parameters. Assume that  $Y$  and  $Z$  are statistically independent. Then the pgf of  $X = Y + Z$  is given by

$$P_X(s) = P_Y(s)P_Z(s) \tag{2.2}$$

$$= \frac{(1 - \rho_1\theta_1 - \rho_2\theta_2)^r}{(1 - \theta_1 - \theta_2)^r - 1} [(1 - \theta_1 s - \theta_2 s^2)^{-r} - 1] (1 - \rho_1\theta_1 s - \rho_2\theta_2 s^2)^{-r} \tag{2.3}$$

in which  $r > 0, \rho_i > 0$  and  $\theta_i > 0$  for  $i = 1, 2$  such that  $\rho_1\theta_1 + \rho_2\theta_2 \leq 1$ . The distribution of a random variable  $X$  with pgf (2.3), we call “the intervened Gegenbauer distribution” or in short “the IGbD”. Clearly, when  $\theta_2 = 0$  the pgf (2.3) reduces to the pgf of the INBD of

(Kumar and Sreeja, 2012) and when  $\rho_2 = 0$  and  $\rho_1 = 0$  the pgf (2.3) reduces to the pgf of the ZTNBD.

Now we obtain the pmf of the IGbD through the following proposition.

**Proposition 2.1.** If  $X$  follows IGbD with pgf as given in (2.3), then the pmf  $g_x$  of  $X$ , for  $x = 1, 2, 3, \dots$  is

$$g_x = \frac{(1 - \rho_1\theta_1 - \rho_2\theta_2)^r}{(1 - \theta_1 - \theta_2)^{-r} - 1} \sum_{i=0}^{x-1} G_{x-i}^r(\theta_1, \theta_2) G_i^r(\rho_1\theta_1, \rho_2\theta_2) \tag{2.4}$$

in which  $r > 0$ ,  $\rho_i > 0$  and  $\theta_i > 0$  for  $i = 1, 2$  such that  $\rho_1\theta_1 + \rho_2\theta_2 \leq 1$  and  $G_n^\lambda(a, b)$  is the Gegenbauer polynomial as defined in (1.3).

*Proof:*

By definition, the pgf  $P_X(s)$  of the IGbD is

$$P_X(s) = \sum_{x=0}^{\infty} s^x g_x \tag{2.5}$$

$$= \frac{(1 - \rho_1\theta_1 - \rho_2\theta_2)^r}{(1 - \theta_1 - \theta_2)^{-r} - 1} [(1 - \theta_1 s - \theta_2 s^2)^{-r} - 1] (1 - \rho_1\theta_1 s - \rho_2\theta_2 s^2)^{-r} \tag{2.6}$$

Applying (1.3) in (2.6), we obtain the following.

$$P_X(s) = \frac{(1 - \rho_1\theta_1 - \rho_2\theta_2)^r}{(1 - \theta_1 - \theta_2)^{-r} - 1} \left\{ \sum_{x=0}^{\infty} \sum_{i=0}^{\infty} G_x^r(\theta_1, \theta_2) G_i^r(\rho_1\theta_1, \rho_2\theta_2) s^{x+i} - \sum_{x=0}^{\infty} G_x^r(\rho_1\theta_1, \rho_2\theta_2) s^x \right\} \tag{2.7}$$

$$= \frac{(1 - \rho_1\theta_1 - \rho_2\theta_2)^r}{(1 - \theta_1 - \theta_2)^{-r} - 1} \left\{ \sum_{x=0}^{\infty} \sum_{i=0}^x G_{x-i}^r(\theta_1, \theta_2) G_i^r(\rho_1\theta_1, \rho_2\theta_2) s^x - \sum_{x=0}^{\infty} G_x^r(\rho_1\theta_1, \rho_2\theta_2) s^x \right\} \tag{2.8}$$

in the light of (1.2).

Now, on equating the coefficients of  $s^x$  in the right hand side expressions of (2.5) and (2.8), we get (2.4).

Next we develop an expression for the  $k^{\text{th}}$  factorial moment of the IGbD through the following proposition.

**Proposition 2.2.** The  $k^{\text{th}}$  factorial moment  $\mu_{[k]}$  of IGbD with pgf given in (2.3) is

$$\mu_{[k]} = k! \left[ G_k^r(\alpha_2, \beta_2) + \delta \sum_{j=0}^{k-1} G_{k-j}^r(\alpha_1, \beta_1) G_j^r(\alpha_2, \beta_2) \right], \tag{2.9}$$

in which

$$\alpha_1 = \frac{\theta_1 + 2\theta_2}{1 - \theta_1 - \theta_2}, \quad \beta_1 = \frac{\theta_2}{1 - \theta_1 - \theta_2}, \quad \alpha_2 = \frac{\rho_1\theta_1 + 2\rho_2\theta_2}{1 - \rho_1\theta_1 - \rho_2\theta_2}, \quad \beta_2 = \frac{\rho_2\theta_2}{1 - \rho_1\theta_1 - \rho_2\theta_2}, \quad \delta = (1 - (1 - \theta_1 - \theta_2)^r)^{-1}$$

and  $G_n^{\lambda}(a, b)$  is as defined in (1.3).

*Proof*

The factorial moment generating function  $H_X(t)$  of the random variable X is

$$H_X(t) = \sum_{k=0}^{\infty} \mu_{[k]} \frac{t^k}{k!} \tag{2.10}$$

By replacing  $s$  by  $1 + t$  in (2.3), we get the  $H_X(t)$  of IGbD as

$$H_X(t) = \frac{(1 - \rho_1\theta_1 - \rho_2\theta_2)^r}{(1 - \theta_1 - \theta_2)^{-r} - 1} [(1 - \theta_1(1+t) - \theta_2(1+t)^2)^{-r} - 1] (1 - \rho_1\theta_1(1+t) - \rho_2\theta_2(1+t)^2)^{-r}$$

On rearranging and substituting  $\alpha_1, \alpha_2, \beta_1, \beta_2$  and  $\delta$ , we get

$$H_X(t) = \delta(1 - \alpha_1 t - \beta_1 t^2)^{-r} (1 - \alpha_2 t - \beta_2 t^2)^{-r} - (\delta - 1)(1 - \alpha_2 t - \beta_2 t^2)^{-r} \tag{2.11}$$

Applying (1.3) in (2.11), we obtain the following.

$$H_X(t) = \delta \sum_{k=0}^{\infty} G_k^r(\alpha_1, \beta_1) t^k \sum_{j=0}^{\infty} G_j^r(\alpha_2, \beta_2) t^j - (\delta - 1) \sum_{k=0}^{\infty} G_k^r(\alpha_2, \beta_2) t^k \tag{2.12}$$

$$= \delta \sum_{k=0}^{\infty} \sum_{j=0}^k G_{k-j}^r(\alpha_1, \beta_1) G_j^r(\alpha_2, \beta_2) t^k - (\delta - 1) \sum_{k=0}^{\infty} G_k^r(\alpha_2, \beta_2) t^k, \tag{2.13}$$

in the light of (1.2).

On equating the coefficient of  $\frac{t^k}{k!}$  on the right hand side expressions of (2.10) and (2.13), we get (2.9).

The mean and variance of the IGbD are respectively

$$\text{Mean} = r(\alpha_2 + \delta\alpha_1),$$

$$\text{Variance} = r(\alpha_2^2 + \delta\alpha_1^2) + r(\alpha_2 + \delta\alpha_1) + 2r(\beta_2 + \delta\beta_1) + r^2\alpha_1^2\delta(1 - \delta),$$

in which  $\alpha_1, \alpha_2, \beta_1, \beta_2$  and  $\delta$  are as given in (2.9).

**Proposition 2.3.** The IGbD is over-dispersed if

$$\alpha_2^2 + \delta\alpha_1^2(r + 1) + 2(\beta_2 + \delta\beta_1) > r\delta^2\alpha_1^2$$

and under-dispersed if

$$\alpha_2^2 + \delta\alpha_1^2(r + 1) + 2(\beta_2 + \delta\beta_1) < r\delta^2\alpha_1^2$$

**Proposition 2.4.** The following is a recurrence relation for probabilities of IGbD, for  $x \geq 1$ ,

$$xg_{x+1} = r\rho_1\theta_1 \sum_{i=0}^x G_i^1(\rho_1\theta_1, \rho_2\theta_2) g_{x-i} + 2r\rho_2\theta_2 \sum_{i=0}^{x-1} G_i^1(\rho_1\theta_1, \rho_2\theta_2) g_{x-i-1} + A(x : r, \rho_1, \theta_1, \rho_2, \theta_2) \tag{2.14}$$

in which

$$A(x : r, \rho_1, \theta_1, \rho_2, \theta_2) = r\theta_1 \sum_{i=0}^x G_{x-i}^r(\rho_1\theta_1, \rho_2\theta_2) G_i^{r+1}(\theta_1, \theta_2) + 2r\theta_2 \sum_{i=0}^{x-1} G_{x-i-1}^r(\rho_1\theta_1, \rho_2\theta_2) G_i^{r+1}(\theta_1, \theta_2)$$

and  $G_n^a(a, b)$  is the Gegenbauer polynomial defined in (1.3).

*Proof*

From (2.3), we have,

$$P_x(s) = \sum_{x=0}^{\infty} s^x g_x \tag{2.15}$$

$$= \frac{(1 - \rho_1\theta_1 - \rho_2\theta_2)^r}{(1 - \theta_1 - \theta_2)^r - 1} [(1 - \theta_1 s - \theta_2 s^2)^{-r} - 1] (1 - \rho_1\theta_1 s - \rho_2\theta_2 s^2)^{-r} \tag{2.16}$$

On differentiating (2.15) and (2.16) with respect to  $s$ , we get the following.

$$\sum_{x=0}^{\infty} (x+1)s^x g_{x+1} = \frac{r(\rho_1\theta_1 + 2\rho_2\theta_2 s)}{(1 - \rho_1\theta_1 s - \rho_2\theta_2 s^2)} P_x(s) + \frac{r(\theta_1 + 2\theta_2 s)}{(1 - \theta_1 s - \rho_2\theta_2 s^2)^{r+1} (1 - \rho_1\theta_1 s - \rho_2\theta_2 s^2)^r} \tag{2.17}$$

By using (1.3) and (2.15) in (2.17), we get

$$\begin{aligned} \sum_{x=0}^{\infty} (x+1) g_{x+1} &= r(\rho_1\theta_1 + 2\rho_2\theta_2 s) \sum_{x=0}^{\infty} g_x \sum_{i=0}^x G_i^1(\rho_1\theta_1, \rho_2\theta_2) s^i + r(\theta_1 + 2\theta_2 s) \sum_{x=0}^{\infty} G_x^r(\rho_1\theta_1, \rho_2\theta_2) s^x \sum_{i=0}^{\infty} G_i^{r+1}(\theta_1, \theta_2) s^i \\ &= r\rho_1\theta_1 \sum_{x=0}^{\infty} \sum_{i=0}^x G_i^1(\rho_1\theta_1, \rho_2\theta_2) g_{x-i} s^x + 2r\rho_2\theta_2 \sum_{x=0}^{\infty} \sum_{i=0}^x G_i^1(\rho_1\theta_1, \rho_2\theta_2) g_{x-i} s^{x+1} \\ &\quad + r\theta_1 \sum_{x=0}^{\infty} \sum_{i=0}^x G_{x-i}^r(\rho_1\theta_1, \rho_2\theta_2) G_i^{r+1}(\theta_1, \theta_2) s^x + 2r\theta_2 \sum_{x=0}^{\infty} \sum_{i=0}^x G_{x-i}^r(\rho_1\theta_1, \rho_2\theta_2) G_i^{r+1}(\theta_1, \theta_2) s^{x+1} \end{aligned} \tag{2.18}$$

in the light of (1.2).

On equating coefficient of  $s^x$  on both sides of the expression (2.18), we obtain (2.14).

### 3. Estimation

Here we discuss the estimation of the parameters of IGbD by the method of maximum likelihood. Let  $a(x)$  be the observed frequency of  $x$  events,  $y$  be the highest value of  $x$  observed. Then the likelihood function of the sample is

$$L = \prod_{x=0}^{\infty} [g_x]^{a(x)}, \tag{3.1}$$

which implies

$$\log L = \sum_{x=1}^{\infty} a(x) \log [g(x)], \tag{3.2}$$

Let  $\hat{r}, \hat{\rho}_1, \hat{\rho}_2, \hat{\theta}_1$  and  $\hat{\theta}_2$  denotes the maximum likelihood estimators of the parameters  $r, \rho_1, \rho_2, \theta_1$  and  $\theta_2$  of the IGbD respectively. Now  $\hat{r}, \hat{\rho}_1, \hat{\rho}_2, \hat{\theta}_1$  and  $\hat{\theta}_2$  are obtained by solving the likelihood equations (3.3), (3.4), (3.5), (3.6) and (3.7), as given below.

$$\frac{\partial \log L}{\partial \theta_1} = 0,$$

implies

$$\sum_{x=1}^y a(x) \left[ \frac{\phi_1(x:r, \theta_1, \theta_2, \rho_1, \rho_2)}{\phi(x:r, \theta_1, \theta_2, \rho_1, \rho_2)} - \frac{r\rho_1\theta_1}{(1-\rho_1\theta_1-\rho_2\theta_2)} + \frac{r\theta_1(1-\theta_1-\theta_2)^{-1}}{1-(1-\theta_1-\theta_2)^r} \right] = 0, \tag{3.3}$$

where

$$\phi(x:r, \theta_1, \theta_2, \rho_1, \rho_2) = \sum_{k=0}^{x-1} G_{x-k}^r(\theta_1, \theta_2) G_k^r(\rho_1\theta_1, \rho_2\theta_2)$$

and

$$\begin{aligned} \phi_1(x:r, \theta_1, \theta_2, \rho_1, \rho_2) = & \sum_{k=0}^{x-1} \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \frac{\Gamma(r+k-i)(\rho_1\theta_1)^{k-2i}(\rho_2\theta_2)^i}{\Gamma(r)\Gamma(i+1)\Gamma(k-2i)} G_{x-k}^r(\theta_1, \theta_2) \\ & + \sum_{k=0}^{x-1} \sum_{j=0}^{\lfloor \frac{x-k}{2} \rfloor} \frac{\Gamma(r+x-k-j)\theta_1^{x-k-2j}\theta_2^j}{\Gamma(r)\Gamma(j+1)\Gamma(x-k-2j)} G_k^r(\rho_1\theta_1, \rho_2\theta_2). \end{aligned}$$

$$\frac{\partial \log L}{\partial \theta_2} = 0, \quad \text{implies}$$

$$\sum_{x=1}^y a(x) \left[ \frac{\phi_2(x:r, \theta_1, \theta_2, \rho_1, \rho_2)}{\phi(x:r, \theta_1, \theta_2, \rho_1, \rho_2)} - \frac{r\rho_2\theta_2}{(1-\rho_1\theta_1-\rho_2\theta_2)} + \frac{r\theta_2(1-\theta_1-\theta_2)^{-1}}{1-(1-\theta_1-\theta_2)^r} \right] = 0 \tag{3.4}$$

where

$$\begin{aligned} \phi_2(x:r, \theta_1, \theta_2, \rho_1, \rho_2) &= \sum_{k=0}^{x-1} \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \frac{\Gamma(r+k-i)(\rho_1\theta_1)^{k-2i}(\rho_2\theta_2)^i}{\Gamma(r)\Gamma(i)\Gamma(k-2i+1)} G_{x-k}^r(\theta_1, \theta_2) \\ &\quad + \sum_{k=0}^{x-1} \sum_{j=0}^{\lfloor \frac{x-k}{2} \rfloor} \frac{\Gamma(r+x-k-j)\theta_1^{x-k-2j}\theta_2^j}{\Gamma(r)\Gamma(j)\Gamma(x-k-2j+1)} G_k^r(\rho_1\theta_1, \rho_2\theta_2) \end{aligned}$$

and  $\phi(x:r, \theta_1, \theta_2, \rho_1, \rho_2)$  is as defined in (3.3).

$$\begin{aligned} \frac{\partial \log L}{\partial \rho_1} &= 0, \quad \text{implies} \\ \sum_{x=1}^y a(x) \left[ \frac{\phi_3(x:r, \theta_1, \theta_2, \rho_1, \rho_2)}{\phi(x:r, \theta_1, \theta_2, \rho_1, \rho_2)} - \frac{r\rho_1\theta_1}{(1-\rho_1\theta_1-\rho_2\theta_2)} \right] &= 0 \end{aligned} \tag{3.5}$$

where

$\phi_3(x:r, \theta_1, \theta_2, \rho_1, \rho_2) = \sum_{k=0}^{x-1} \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \frac{\Gamma(r+k-i)(\rho_1\theta_1)^{k-2i}(\rho_2\theta_2)^i}{\Gamma(r)\Gamma(i+1)\Gamma(k-2i)} G_{x-k}^r(\theta_1, \theta_2)$  and  $\phi(x:r, \theta_1, \theta_2, \rho_1, \rho_2)$  is as defined in (3.3).

$$\begin{aligned} \frac{\partial \log L}{\partial \rho_2} &= 0, \quad \text{implies} \\ \sum_{x=1}^y a(x) \left[ \frac{\phi_4(x:r, \theta_1, \theta_2, \rho_1, \rho_2)}{\phi(x:r, \theta_1, \theta_2, \rho_1, \rho_2)} - \frac{r\rho_2\theta_2}{(1-\rho_1\theta_1-\rho_2\theta_2)} \right] &= 0 \end{aligned} \tag{3.6}$$

where

$\phi_4(x:r, \theta_1, \theta_2, \rho_1, \rho_2) = \sum_{k=0}^{x-1} \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \frac{\Gamma(r+k-i)(\rho_1\theta_1)^{k-2i}(\rho_2\theta_2)^i}{\Gamma(r)\Gamma(i)\Gamma(k-2i+1)} G_{x-k}^r(\theta_1, \theta_2)$  and  $\phi(x:r, \theta_1, \theta_2, \rho_1, \rho_2)$  is as defined in (3.3).

$$\begin{aligned} \frac{\partial \log L}{\partial r} &= 0, \quad \text{implies} \\ \sum_{x=1}^y a(x) \left[ \frac{\phi_5(x:r, \theta_1, \theta_2, \rho_1, \rho_2)}{\phi(x:r, \theta_1, \theta_2, \rho_1, \rho_2)} - \frac{\log(1-\theta_1-\theta_2)}{1-(1-\theta_1-\theta_2)^r} + \log(1-\rho_1\theta_1-\rho_2\theta_2) \right] &= 0 \end{aligned} \tag{3.7}$$

where

$\phi_5(x:r, \theta_1, \theta_2, \rho_1, \rho_2) = \sum_{k=0}^{x-1} \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \frac{\Gamma(r+k-i)(\rho_1\theta_1)^{k-2i}(\rho_2\theta_2)^i}{\Gamma(r)\Gamma(i+1)\Gamma(k-2i+1)} G_{x-k}^r(\theta_1, \theta_2)$  and  $\phi(x:r, \theta_1, \theta_2, \rho_1, \rho_2)$  is as defined in (3.3).



Note that these likelihood equations do not always have closed form solutions. Therefore, maximum of the likelihood function is attained at the border of domain of the parameters. We obtain the second order partial derivatives of  $\log L$  with respect to the parameters  $r, \theta_1, \theta_2, \rho_1$  and  $\rho_2$  by using MATHEMATICA software's and observed that these equations gives negative values for  $r > 0, \theta_1 > 0, \theta_2 > 0, \rho_1 > 0$  and  $\rho_2 > 0$ .

In order to illustrate the usefulness of the model IGbD, we have considered three sets of real life data and compared with some existing competing intervened type models such as IGD, INBD and MINBD along with ZTNBD and ZTGbD. The numerical results obtained are summarized in Table 1, 2 and 3. From the tables it is clear that the IGbD gives better fit to all those data sets compared to ZTGbD, IGD, ZTNBD, INBD and MINBD. The data given in Table 1 and 2 are related to the number of research publications in the credit of authors. The authors have more number of publications to their credit, that may be due to the influence of some interventions such as measures taken for academic grade, promotion, research incentives etc. The data given in Table 3 is related to the number of fly eggs on flower heads. The data indicates that there are flower heads with more number of eggs which might be due to the effect of certain interventions such as nutritious food, rainfall, temperature, humidity, sunlight etc.

#### 4. Simulation

For examining the performance of the maximum likelihood estimators we have simulated data sets corresponding to the following set of parameters, by using Markov Chain Monte-Carlo simulation procedure.

Parameter set I:  $r = 1.33, \rho_1 = 0.356, \rho_2 = 0.285, \theta_1 = 0.263, \theta_2 = 0.17$  (over-dispersion)

Parameter set II:  $r = 0.33, \rho_1 = 1.356, \rho_2 = 1.285, \theta_1 = 0.063, \theta_2 = 0.17$  (under-dispersion).

We have computed the bias and mean square errors in each case and presented in Table 4 and 5. From the tables it can be seen that as the sample size increases, the absolute bias and mean square error decreases.

In order to know the effectiveness of the intervention parameter, we have carried out the test  $H_0: \rho_1 = 0, \rho_2 = 0$  vs  $H_1: \rho_1 > 0, \rho_2 > 0$  by using generalized likelihood ratio procedure to the simulated data sets. For the data set generated by the parameter set I, the probability of Type I error is 0.00018 and that of the data set generated by the parameter set II is 0.00026.

**Table 1. Data represents the distribution of number of articles on theoretical Statistics and Probability for years 1940-49 and initial letter N-R of the author's name. For details, see (Kendal M G,1961).**

|                   |     | Expected frequencies      |                          |                         |                       |                           |                                         |
|-------------------|-----|---------------------------|--------------------------|-------------------------|-----------------------|---------------------------|-----------------------------------------|
| x                 | f   | ZTGbD                     | IGD                      | ZTNBD                   | INBD                  | MINBD                     | IGbD                                    |
| 1                 | 83  | 42.18                     | 61.18                    | 76.08                   | 98.709                | 79.315                    | 80.12                                   |
| 2                 | 18  | 35.01                     | 38.10                    | 29.79                   | 21.32                 | 23.063                    | 22.98                                   |
| 3                 | 13  | 17.75                     | 20.28                    | 15.12                   | 6.003                 | 17.538                    | 13.434                                  |
| 4                 | 9   | 12.69                     | 11.14                    | 8.55                    | 5.775                 | 7.77                      | 8.069                                   |
| 5                 | 7   | 8.65                      | 5.95                     | 5.13                    | 5.154                 | 5.929                     | 5.665                                   |
| 6                 | 7   | 6.35                      | 3.17                     | 3.18                    | 5.031                 | 3.194                     | 5.425                                   |
| 7                 | 2   | 4.69                      | 1.69                     | 2.48                    | 0.008                 | 2.377                     | 2.323                                   |
| 8                 | 5   | 16.68                     | 2.46                     | 3.72                    | 2.00                  | 4.814                     | 5.984                                   |
| Total             | 144 | 144                       | 144                      | 144                     | 144                   | 144                       | 144                                     |
| df                |     | 3                         | 3                        | 3                       | 2                     | 1                         | 1                                       |
| Estimated         |     | $\hat{i} = 0.1224$        | $\hat{\rho} = 5.96604$   | $\hat{i} = 0.0594$      | $\hat{i} = 0.23$      | $\hat{i} = 0.286212$      | $\hat{i} = 0.404546$                    |
| Value of          |     | $\hat{\theta}_1 = 0.5195$ | $\hat{\theta} = 0.08941$ | $\hat{\theta} = 0.7392$ | $\hat{\rho} = 0.75$   | $\hat{\rho}_1 = 6.61193$  | $\hat{\rho}_1 = 27.255$                 |
| parameters        |     | $\hat{\theta}_2 = 0.2796$ |                          |                         | $\hat{\theta} = 0.23$ | $\hat{\rho}_2 = 2.82871$  | $\hat{\rho}_2 = 17.6079$                |
|                   |     |                           |                          |                         |                       | $\hat{\theta} = 0.114679$ | $\hat{\theta}_1 = 0.0276485$            |
|                   |     |                           |                          |                         |                       |                           | $\hat{\theta}_2 = 5.86 \times 10^{-12}$ |
| Chi- square value |     | 60.157                    | 29.537                   | 9.538                   | 26.816                | 4.103                     | 2.282                                   |
| P - value         |     | $5.44 \times 10^{-13}$    | $1.727 \times 10^{-6}$   | 0.049                   | 0.00015               | 0.043                     | 0.892                                   |
| AIC               |     | 492.68                    | 456.804                  | 444.462                 | 547.858               | 448.74                    | 442.952                                 |
| BIC               |     | 496.62                    | 462.744                  | 444.778                 | 548.027               | 446.68                    | 443.801                                 |
| AICc              |     | 486.84                    | 452.888                  | 444.54                  | 556.767               | 445.024                   | 441.381                                 |

**Table 2. Data represents the Article by journal for years 1950-1958 and initial letters N-R of authors name. For details see (Kendal M G,1961)**

| Expected frequencies |                           |                          |                          |                         |                         |                                  |        |
|----------------------|---------------------------|--------------------------|--------------------------|-------------------------|-------------------------|----------------------------------|--------|
| x                    | f                         | ZTGbD                    | IGD                      | ZTNBD                   | INBD                    | MINBD                            | IGbD   |
| 1                    | 198                       | 108.85                   | 152.06                   | 186.94                  | 194.97                  | 196.49                           | 195.34 |
| 2                    | 60                        | 73.07                    | 89.34                    | 70.69                   | 63.12                   | 54.05                            | 61.25  |
| 3                    | 20                        | 49.18                    | 48.38                    | 35.32                   | 31.72                   | 41.91                            | 22.21  |
| 4                    | 21                        | 33.89                    | 26.01                    | 19.79                   | 18.78                   | 15.01                            | 19.29  |
| 5                    | 10                        | 23.61                    | 13.97                    | 11.81                   | 11.92                   | 14.86                            | 12.24  |
| 6                    | 5                         | 16.58                    | 7.50                     | 7.34                    | 7.85                    | 5.86                             | 8.042  |
| 7                    | 32                        | 38.82                    | 8.71                     | 14.08                   | 17.61                   | 17.79                            | 27.60  |
| Total                | 346                       | 346                      | 346                      | 346                     | 346                     | 346                              | 346    |
| df                   |                           | 3                        | 4                        | 4                       | 3                       | 2                                | 1      |
| Estimated            | $\hat{i} = 0.7685$        | $\hat{\rho} = 10.6692$   | $\hat{i} = 0.018126$     | $\hat{i} = 0.3876$      | $\hat{i} = 0.187806$    | $\hat{i} = 0.419486$             |        |
| Value of             | $\hat{\theta}_1 = 0.6332$ | $\hat{\theta} = 0.05035$ | $\hat{\theta} = 0.74281$ | $\hat{\rho} = 16.174$   | $\hat{\rho}_1 = 2.982$  | $\hat{\rho}_1 = 273.894$         |        |
| parameters           | $\hat{\theta}_2 = 0.0705$ |                          |                          | $\hat{\theta} = 0.0465$ | $\hat{\rho}_2 = 2.532$  | $\hat{\rho}_2 = 646.20$          |        |
|                      |                           |                          |                          |                         | $\hat{\theta} = 0.2384$ | $\hat{\theta}_1 = 0.002712$      |        |
|                      |                           |                          |                          |                         |                         | $\hat{\theta}_2 = 7.93*10^{-11}$ |        |
| Chi- square value    | 114.701                   | 105.32                   | 32.801                   | 17.891                  | 27.578                  | 2.697                            |        |
| P - value            | 0.00                      | 0.00                     | $1.312*10^{-6}$          | 0.006                   | $1.027 *10^{-6}$        | 0.846                            |        |
| AIC                  | 1127                      | 1067                     | 1044                     | 1044                    | 1042                    | 1031                             |        |
| BIC                  | 1133                      | 1075                     | 1045                     | 1055                    | 1045                    | 1036                             |        |
| AICc                 | 1121                      | 1063                     | 1044                     | 1044                    | 1044                    | 1031                             |        |

**Table 3. Data representing the counts of flower heads with 1, 2,...,9 fly eggs, (Finney and Varley,1955).**

| Expected frequencies       |    |                                                                              |                                                  |                                             |                                                                       |                                                                                                      |                                                                                                                                  |
|----------------------------|----|------------------------------------------------------------------------------|--------------------------------------------------|---------------------------------------------|-----------------------------------------------------------------------|------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------|
| X                          | f  | ZTGbD                                                                        | IGD                                              | ZTNBD                                       | INBD                                                                  | MINBD                                                                                                | IGbD                                                                                                                             |
| 1                          | 22 | 31.38                                                                        | 29.158                                           | 10.593                                      | 19.324                                                                | 34.353                                                                                               | 22.695                                                                                                                           |
| 2                          | 18 | 20.04                                                                        | 24.373                                           | 15.627                                      | 22.562                                                                | 17.462                                                                                               | 18.802                                                                                                                           |
| 3                          | 18 | 11.66                                                                        | 15.497                                           | 16.87                                       | 17.821                                                                | 18.613                                                                                               | 17.473                                                                                                                           |
| 4                          | 11 | 7.61                                                                         | 8.878                                            | 14.876                                      | 11.919                                                                | 7.261                                                                                                | 11.59                                                                                                                            |
| 5                          | 9  | 5.09                                                                         | 4.829                                            | 11.352                                      | 7.274                                                                 | 5.311                                                                                                | 7.275                                                                                                                            |
| 6                          | 6  | 3.49                                                                         | 2.552                                            | 7.765                                       | 4.191                                                                 | 2.07                                                                                                 | 5.04                                                                                                                             |
| 7                          | 3  | 2.44                                                                         | 1.326                                            | 4.872                                       | 2.32                                                                  | 1.025                                                                                                | 2.544                                                                                                                            |
| 8                          | 0  | 1.72                                                                         | 0.681                                            | 2.851                                       | 1.248                                                                 | 0.366                                                                                                | 1.865                                                                                                                            |
| 9                          | 1  | 3.57                                                                         | 0.706                                            | 3.194                                       | 1.341                                                                 | 1.593                                                                                                | 0.716                                                                                                                            |
| Total                      | 88 | 88                                                                           | 88                                               | 88                                          | 88                                                                    | 88                                                                                                   | 88                                                                                                                               |
| df                         |    | 2                                                                            | 2                                                | 4                                           | 2                                                                     | 1                                                                                                    | 1                                                                                                                                |
| Estimated<br>of parameters |    | $\hat{i} = 0.3437$<br>$\hat{\theta}_1 = 0.5824$<br>$\hat{\theta}_2 = 0.1440$ | $\hat{\rho} = 0.6578$<br>$\hat{\theta} = 0.5042$ | $\hat{i} = 9.23$<br>$\hat{\theta} = 0.2884$ | $\hat{i} = 2.3601$<br>$\hat{\rho} = 6.705$<br>$\hat{\theta} = 0.0667$ | $\hat{i} = 10.503$<br>$\hat{\rho}_1 = 0.00586$<br>$\hat{\rho}_2 = 0.3960$<br>$\hat{\theta} = 0.0874$ | $\hat{i} = 2.615$<br>$\hat{\rho}_1 = 0.2841$<br>$\hat{\rho}_2 = 3.015$<br>$\hat{\theta}_1 = 0.2746$<br>$\hat{\theta}_2 = 0.0351$ |
| Chi -square value          |    | 11.376                                                                       | 12.193                                           | 18.252                                      | 3.415                                                                 | 13.967                                                                                               | 0.68                                                                                                                             |
| P - value                  |    | $3.38 * 10^{-3}$                                                             | $2.25 * 10^{-3}$                                 | $1.10 * 10^{-3}$                            | 0.755                                                                 | $1.86 * 10^{-4}$                                                                                     | 0.995                                                                                                                            |
| AIC                        |    | 351.24                                                                       | 342.778                                          | 348.164                                     | 346.19                                                                | 350.00                                                                                               | 338.63                                                                                                                           |
| BIC                        |    | 354.195                                                                      | 347.733                                          | 348.052                                     | 353.62                                                                | 350.95                                                                                               | 341.02                                                                                                                           |
| AICc                       |    | 345.519                                                                      | 338.916                                          | 348.30                                      | 346.47                                                                | 342.47                                                                                               | 339.34                                                                                                                           |

**Table 4. Bias and mean square errors (within brackets) of estimators of the parameters of IGbD using simulated data for parameter set I**

| Parameters | Sample size 500       | Sample size 1000      |
|------------|-----------------------|-----------------------|
| $r$        | -0.04415<br>(0.00274) | -0.01245<br>(0.00015) |
| $\rho_1$   | 0.262<br>(0.07656)    | 0.136<br>(0.01855)    |
| $\rho_2$   | 0.215<br>(0.04728)    | 0.105<br>(0.01102)    |
| $\theta_1$ | -0.077<br>(0.00596)   | -0.0325<br>(0.00105)  |
| $\theta_2$ | -0.0314<br>(0.00098)  | -0.0124<br>(0.00015)  |

**Table 5. Bias and mean square errors (within brackets) of estimators of the parameters of IGbD using simulated data for parameter set II**

| parameters | Sample size 500       | Sample size 1000       |
|------------|-----------------------|------------------------|
| $r$        | -0.04039<br>(0.00163) | -0.01026<br>(0.000105) |
| $\rho_1$   | -0.08959<br>(0.00810) | -0.06215<br>(0.00386)  |
| $\rho_2$   | -0.18102<br>(0.03405) | -0.0916<br>(0.00839)   |
| $\theta_1$ | 0.00619<br>(0.00003)  | 0.00235<br>(0.0000059) |
| $\theta_2$ | 0.01748<br>(0.00030)  | 0.00961<br>(0.000092)  |

## 5. Summary and Conclusion

A new class of the intervened type distribution is introduced as a generalization of both the zero truncated negative binomial distribution and the intervened negative distribution through the name intervened Gegenbaur distribution. We obtained several important properties of the distribution and fitted the model to three real life data sets. A brief simulation study is carried out for assessing the performance of the maximum likelihood estimators of the parameters of the distribution and to examine the significance of the intervention parameters.

## Acknowledgement

The authors are highly grateful to the Editor, the Associate Editor and the anonymous referees for their fruitful suggestions, those greatly improved the quality of the paper.

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