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New Exponential-Type Estimators of Finite Population Variance Using Auxiliary Information

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ABSTRACT

In this study, some existing finite population variance estimators for study variable have been modified using linear combination and power improvement techniques. Asymptotic properties (Biases and MSEs) of the suggested estimators are derived up to the terms of first-order approximation using Taylor's Series expansion. The numerical illustration was also supported by six real-life data sets and simulated data set using R to corroborate the theoretical results. In general, the results reveal that the proposed estimators outperformed the existing estimators considered in the study.

Keywords: Exponential, Efficiency, Estimator, Mean Square Error, Bias

1. Introduction

Sampling is the method of selecting a representative subset of the population called sample. Sampling makes research more accurate and economical. It's the sampling method which actually determines the generalizability of the research findings. In simple words, the process of choosing a sample of the population to study is called sampling. The problem of estimating the finite population mean in the presence of an auxiliary variable has been widely discussed in the finite population sampling literature. However, in practice, the problem of estimating the variance of the population also becomes important. Like estimators of the finite population mean, few authors including (Das and Tripathi 1978); (Isaki, 1983); (Srivastava and Jhajj, 1980); (Biradar and Singh, 1988); (Singh et al., 1988, 2003); (Kadilar and Cingi, 2007, 2006, 2005); (Grover, 2007, 2010); (Shabbir and Gupta, 2007); (Singh and Vishwakarma, 2008); (Singh and Solanki, 2013a, 2013b,); (Solanki and Singh, 2013), (Ishaq *et al.*, 2020), (Muili *et al.*, 2019) and others have turned their attention to estimating population variance in the presence of auxiliary information.

Consider the finite population $\Omega:\{\Omega_1,\Omega_2,...,\Omega_N\}$ made up of N units. Let Y and X denote the study and the auxiliary variables taking the values y_h and x_h respectively on the h^{th} units Ω_h of the population Ω and sample of size n drawn by simple random sampling without replacement ($SRSWOR$) from the population Ω. We assume that the size of the population N is large with respect to n, so that the finite population correction term is neglected.

The conventional unbiased estimator of variance is defined as;

$$t_{s_y^2}=\frac{1}{n-1}\sum_{i=1}^{n}\left(y_i-\bar{y}\right)^2$$

$$\text{var}\left(t_{s_y^2}\right)=S_Y^4\mu_{40}$$

Ratio estimator of the population variance $s^2{}_y$ due to $Isaki$ (1983) is given by

$$t_{Isaki}=s_y^2\left(\frac{S_x^2}{s_x^2}\right),$$

$$MSE\left(t_{Isaki}\right)\cong S_Y^4\left[\mu_{40}+\mu_{04}-2\mu_{22}\right]$$

Isaki(1983) estimator is given as;

$$t_{Isaki}^{\text{Re}g}=s_y^2+b_{(s_y^2s_x^2)}(S_x^2-s_x^2),$$

$$MSE\left(t_{Isaki}^{\text{Re}g}\right)\cong S_y^4\mu_{40}\left(1-\rho_{s_y^2s_x^2}^2\right)$$

Where $\rho_{s_y^2s_x^2}=\frac{\mu_{22}}{\sqrt{\mu_{40}\mu_{04}}}$ is the population correlation coefficient between y and x

Singh *et al*. (1988) estimator is given as;

$$t_{sh} = \phi_1 s_y^2 + \phi_2 (S_X^2 - s_x^2)$$

$$MSE(t_{sh})_{\min} \cong \frac{S_y^4 \mu_{40}(1-\rho^2_{s_y^2 s_x^2})}{1+\mu_{40}(1-\rho^2_{s_y^2 s_x^2})}$$

Where ϕ_1 and ϕ_2 are unknown constants to be determined

$$\phi_{1_{opt}} = \frac{\mu_{04}}{\left(\mu_{04} + \mu_{40}\mu_{04} - \mu_{22}^2\right)} \text{ and } \phi_{1_{opt}} = \frac{S_x^2}{S_y^2}\frac{\mu_{22}}{\left(\mu_{04} + \mu_{40}\mu_{04} - \mu_{22}^2\right)}$$

Bahl and Tuteja (1991) ratio type exponential estimator for the population variance is given by

$$t_{BT} = S_y^2 \exp\left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right)$$

$$MSE(t_{BT}) \cong S_y^4\left[\mu_{40} + \frac{1}{4}\mu_{04} - \mu_{22}\right].$$

Shabbir and Gupta (2007) proposed the following estimator for c is given by

$$t_{SG} = \left[\phi_3 s_y^2 + \phi_4 (S_x^2 - s_x^2)\right]\exp\left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right)$$

$$MSE(t_{SG})_{\min} \cong \frac{S_y^4}{64}\left(\frac{-\mu_{04}^2 - 16\mu_{40}(1-\rho^2_{(s_y^2 s_x^2)})(\mu_{04}-4)}{1+\mu_{40}(1-\rho^2_{(s_y^2 s_x^2)})}\right)$$

Where $\phi_{3_{opt}} = \frac{\mu_{04}}{8}\left(\frac{8-\mu_{04}}{\mu_{04} + \mu_{40}\mu_{04} - \mu_{22}^2}\right)$ and

$$\phi_{4_{opt}} = \frac{S_x^2}{8S_y^2}\left(\frac{-4\mu_{04} + \mu_{04} + 8\mu_{22} - \mu_{22}\mu_{04} + 4\mu_{40}\mu_{04} - 4\mu_{22}^2}{\left(\mu_{04} + \mu_{40}\mu_{04} - \mu_{22}^2\right)}\right)$$

Yadav and Kadilar (2014) proposed estimator for S_y^2 as;

$$t_{YK} = s_y^2\left[\lambda\left\{\frac{(1-\varphi)s_x^2 + \varphi S_x^2}{\varphi s_x^2 + (1-\varphi)S_x^2}\right\} + (1-\lambda)\left\{\frac{\varphi s_x^2 + (1-\varphi)S_x^2}{(1-\varphi)s_x^2 + \varphi S_x^2}\right\}\right],$$

$$MSE(t_{YK})_{\min} \cong S_y^4 \mu_{40}\left(1-\rho^2_{(S_y^2 s_y^2)}\right)$$

Where λ and φ are appropriate chosen constants and $\left(\lambda_{opt}, \varphi_{opt}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$

Singh and Malik (2014) estimator for S_y^2, is given by

$$t_{SM} = s_y^2\left[\phi_7 + \phi_8(S_x^2 - s_x^2)\right]\exp\left\{\Omega\frac{m(S_x^2 - s_x^2)}{m(S_x^2 + s_x^2) + 2\eta}\right\}$$

$$MSE(t_{SM})_{\min} \cong \frac{S_y^4}{64}\left[\frac{\mu_{04}\left\{\mu_{04}(\mu_{04}+8\mu_{22}) + 16(\mu_{04}-4)MSE(t_{Isaki}^{\mathrm{Reg}}) + 16\mu_{22}(\mu_{22}-\mu_{04})\right\}}{-\mu_{04}(1+\mu_{40}+2\mu_{22}) + 4\mu_{22}^2}\right]$$

.

Where $m = 1, \eta = 0, \phi_{7_{opt}} = \frac{1}{4}\left(\frac{-12\mu_{04}\mu_{22} + 3\mu_{04}^2 + 16\mu_{22}^2 - 8\mu_{04} - 2\mu_{04}^2}{\mu_{04}^2 - 4\mu_{04}\mu_{22} + 8\mu_{22}^2 - 2\mu_{40}\mu_{04} - 2\mu_{04} - \mu_{04}^2}\right)$

and

$$\phi_{8_{opt}} = \frac{1}{4S_x^2}\left(\frac{-6\mu_{04}\mu_{22} + \mu_{04}^2 + 8\mu_{22}^2 - 4\mu_{04} + 8\mu_{22} - 8\mu_{04}\mu_{22} + 4\mu_{04}\mu_{40}}{\mu_{04}^2 - 4\mu_{04}\mu_{22} + 8\mu_{22}^2 - 2\mu_{40}\mu_{04} - 2\mu_{04} - \mu_{04}^2}\right)$$

Singh *et al.* (2014) proposed the following estimators for S_y^2, as follows;

$$t_{sh_1}^{AM} = \left(\frac{s_y^2}{2}\right)\left(1 + \frac{S_x^2}{s_x^2}\right)$$

$$t_{sh_1}^{GM} = s_y^2\left(\frac{S_x}{s_x}\right)$$

$$t_{sh_1}^{HM} = \frac{2s_y^2}{\left(1 + \frac{S_x^2}{s_x^2}\right)}$$

$$t_{sh_2}^{AM} = \left(\frac{s_y^2}{2}\right)\left(1 + \exp\left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right)\right)$$

$$t_{sh_2}^{GM} = s_y^2\exp\left(\frac{1}{2}\left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right)\right)$$

$$t_{sh_2}^{HM} = \frac{2s_y^2}{\left(1 + \exp\left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right)\right)}$$

$$t_{sh_3}^{AM} = \left(\frac{s_y^2}{2}\right)\left(\frac{S_x^2}{s_x^2} + \exp\left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right)\right)$$

$$t_{sh_3}^{GM} = s_y^2\left(\frac{S_x}{s_x}\right)\exp\left(\frac{1}{2}\left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right)\right)$$

$$t_{sh_3}^{HM} = \frac{2s_y^2}{\left(\frac{s_x^2}{S_x^2} + \exp\left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right)\right)}$$

$$MSE\left(t_{sh_1}^{AM}\right) = MSE\left(t_{sh_1}^{GM}\right) = MSE\left(t_{sh_1}^{HM}\right) = MSE\left(t_{sh_4}^{AM}\right) = MSE\left(t_{sh_4}^{GM}\right) = MSE\left(t_{sh_4}^{HM}\right)$$
$$= \frac{S_y^4}{n}\left[(\mu_{40} - 1) + \frac{(\mu_{04} - 1)}{4}(1 - 4c)\right]$$

$$MSE\left(t_{sh_2}^{AM}\right) = MSE\left(t_{sh_2}^{GM}\right) = MSE\left(t_{sh_2}^{HM}\right) = \frac{S_y^4}{n}\left[(\mu_{40} - 1) + \frac{(\mu_{04} - 1)}{16}(1 - 8c)\right]$$

$$MSE\left(t_{sh_3}^{AM}\right) = MSE\left(t_{sh_3}^{GM}\right) = MSE\left(t_{sh_3}^{HM}\right) = \frac{S_y^4}{n}\left[(\mu_{40} - 1) + \frac{3(\mu_{04} - 1)}{16}(3 - 8c)\right]$$

Where $c = \frac{(\mu_{22} - 1)}{(\mu_{04} - 1)}$

Swain (2015) estimator for S_y^2 is given by

$$t_{Swain} = S_y^2\left[\pi\left(\frac{S_x^2}{s_x^2}\right)^{\alpha_1} + (1-\pi)\left(\frac{s_x^2}{S_x^2}\right)^{\alpha_2}\right]^{\kappa}$$

$$MSE(t_{Swain})_{\min} \cong S_y^4\mu_{40}\left(1 - \rho^2_{(s_y^2 s_x^2)}\right)$$

Where π, α_1, α_2 and $k = (1, -1)$ to be chosen suitably. And $\pi_{opt} = \frac{\left(k\alpha_2 + \frac{\mu_{22}}{\mu_{04}}\right)}{k(\alpha_1 + \alpha_2)}$

Yaqub and Shabbir (2016) proposed an improved class of estimator for S_y^2, is given by

$$t_{YS} = s_y^2\left[\phi_9 + \phi_{10}(S_x^2 - s_x^2)\right]\left(\frac{mS_x^2 + \eta}{ms_x^2 + \eta}\right)\left[\frac{1}{2}\exp\left\{\frac{m(S_x^2 - s_x^2)}{m(S_x^2 + s_x^2) + 2\eta}\right\} + \frac{1}{2}\exp\left\{\frac{m(s_x^2 - S_x^2)}{m(s_x^2 + S_x^2) + 2\eta}\right\}\right]$$

$$MSE(t_{YS})_{\min} \cong \frac{S_y^4}{16}\left(\frac{64(1-\mu_{04})S_y^{-4}MSE(t_{Isaki}^{\mathrm{Re}g})-\mu_{04}^2}{\mu_{04}+4(1-\mu_{04})+4S_y^{-4}MSE(t_{Isaki}^{\mathrm{Re}g})}\right)$$

Where $\phi_{9_{opt}} = \frac{\mu_{04}}{2}\left(\frac{1+7\left(1-\mu_{04}\right)}{\mu_{04}^2+4\mu_{04}\left(1-\mu_{04}\right)+4\mu_{40}\mu_{04}-4\mu_{22}^2}\right)$ and

$$\phi_{10_{opt}} = \frac{S_y^2}{2S_x^2}\left(\frac{\mu_{22}+7\mu_{22}\left(1-\mu_{04}\right)-8\mu_{04}\left(1-\mu_{04}\right)+8\mu_{40}\mu_{04}-8\mu_{22}^2}{\mu_{04}^2+4\mu_{04}\left(1-\mu_{04}\right)+4\mu_{40}\mu_{04}-4\mu_{22}^2}\right)$$

Singh *et al*. (2018) proposed the following chain ratio-ratio-type exponential estimators of finite population variance for S_y^2, as follows;

$$t_{SH}^{CR} = t_R\left(\frac{S_x^2}{s_x^2}\right)$$

$$t_{SH}^{C\mathrm{Re}} = s_y^2\exp\left(2\left(\frac{S_x^2-s_x^2}{S_x^2+s_x^2}\right)\right)$$

$$t_{SH}^{CR\mathrm{Re}} = s_y^2\left(\frac{mS_x^2+\eta}{ms_x^2+\eta}\right)^{\beta}\exp\left(\frac{\lambda m\left(S_x^2-s_x^2\right)}{m\left(S_x^2+s_x^2\right)+2\eta}\right)$$

$$MSE\left(t_{SH}^{CR}\right) = \frac{S_y^4}{n}\left[\left(\mu_{40}-1\right)+4\left(\mu_{04}-1\right)\left(1-c\right)\right]$$

$$MSE\left(t_{SH}^{C\mathrm{Re}}\right) = \frac{S_y^4}{n}\left[\left(\mu_{40}-1\right)+\left(\mu_{04}-1\right)\left(1-2c\right)\right]$$

$$MSE\left(t_{SH}^{CR\mathrm{Re}}\right) = \frac{S_y^4}{n}\left[\left(\mu_{40}-1\right)-\left(\mu_{04}-1\right)c^2\right]$$

Where $\left(\beta,\lambda\right) = \left(\frac{1}{2},\frac{1}{2}\right),\left(1,1\right)$

Muneer *et al*. (2018) estimator for S_y^2, is given by

$$t_{muneer} = s_y^2\left[\phi_{11}\left(\frac{S_x^2}{s_x^2}\right)+\phi_{12}\left(\frac{s_x^2}{S_x^2}\right)\right]\exp\left(\frac{S_x^2-s_x^2}{S_x^2+s_x^2}\right)$$

$$MSE(t_{Muneer})_{\min} \cong \frac{S_y^4}{64}\left[\frac{\begin{array}{c}64\mu_{22}^2\mu_{40} - 48\mu_{22}^2\mu_{04} - 128\mu_{22}\mu_{40}\mu_{04} + 48\mu_{22}\mu_{04}^2 \\ + 64\mu_{40}\mu_{04}^2 + 9\mu_{04}^3 + 64\mu_{22}^2 - 64\mu_{40}\mu_{04}\end{array}}{16\mu_{22}^2 - 16\mu_{22}\mu_{04} - 4\mu_{40}\mu_{04} + \mu_{04}^2 - 4\mu_{04}}\right].$$

$$\phi_{11_{opt}} = \frac{1}{8}\left(\frac{16\mu_{22}^2 + 16\mu_{22}\mu_{20} - 24\mu_{22}\mu_{04} - 16\mu_{40}\mu_{04} - \mu_{04}^2 - 16\mu_{22} - 8\mu_{04}}{16\mu_{22}^2 - 16\mu_{22}\mu_{04} - 4\mu_{40}\mu_{04} + \mu_{04}^2 - 4\mu_{04}}\right)$$

and

$$\phi_{12_{opt}} = \frac{1}{8}\left(\frac{48\mu_{22}^2 - 16\mu_{22}\mu_{40} - 72\mu_{22}\mu_{04} + 16\mu_{40}\mu_{02} + 21\mu_{04}^2 + 16\mu_{22} - 24\mu_{04}}{16\mu_{22}^2 - 16\mu_{22}\mu_{02} - 4\mu_{20}\mu_{02} + \mu_{02}^2 - 4\mu_{02}}\right)$$

2. Proposed Estimators

We proposed the following estimators of finite population variance.

$$t_1^{OM} = s^2{}_y\left(\frac{S^2{}_X}{s^2{}_x}\right)^{\Psi}\exp\left(\frac{S^2{}_X - s^2{}_x}{S^2{}_X + s^2{}_x}\right) \tag{1}$$

$$t_2^{OM} = s^2{}_y\left[\frac{(1+\Delta)S^2{}_X + (1-\Delta)s^2{}_x}{(1-\Delta)S^2{}_X + (1+\Delta)s^2{}_x}\right]\exp\left(\frac{S^2{}_X - s^2{}_x}{S^2{}_X + s^2{}_x}\right) \tag{2}$$

$$t_3^{OM} = S^2{}_y\left[\frac{S^2{}_X}{\Theta s^2{}_x + (1-\Theta)S^2{}_X}\right]\exp\left(\frac{S^2{}_X - s^2{}_x}{S^2{}_X + s^2{}_x}\right) \tag{3}$$

Where Δ, Ψ and Θ are constants to be estimated to minimize the MSE of t_1^{OM} , t_2^{OM} and t_3^{OM} respectively.

2.1 Properties of the Proposed Estimator t_1^{OM}

To obtain the MSE expression of the estimator to the first degree of approximation, we defined the following matrix relations;

$\delta_y = \frac{s_y^2 - S_y^2}{S_y^2}, \delta_x = \frac{s_x^2 - S_x^2}{S_x^2}$, Such that; $E(\delta_y) = E(\delta_x) = 0$,

$E(\delta_y^2) = \theta(\beta_{2(y)} - 1) = \mu_{40,}$ $E(\delta_x^2) = \theta(\beta_{2(x)} - 1) = \mu_{04,}$

$E(\delta_y\delta_x) = \theta(\lambda_{22} - 1) = \mu_{22,}$ $\theta = \frac{1}{n} - \frac{1}{N}$. And $\beta_{2(y)}$ and $\beta_{2(x)}$ denote coefficient of kurtosis of y and x respectively

The MSE is given as;

$$MSE = d\Sigma d' \tag{4}$$

Using (4), the MSE of t_1^{OM} can be obtained as;

$$MSE(t_1^{OM}) = d\Sigma d' \tag{5}$$

Where $d = \left[\frac{d}{ds_y^2} | S_X^2 S_Y^2 \quad \frac{d}{ds_x^2} | S_X^2 S_Y^2 \right]$, $\Sigma = \begin{bmatrix} \mathrm{var}(s_y^2) & \mathrm{cov}(s_x^2 s_y^2) \\ \mathrm{cov}(s_x^2 s_y^2) & \mathrm{var}(s_x^2) \end{bmatrix}$ and $d' =$ transpose of d

Now,

$$\frac{d}{ds_y^2} / S^2{}_X S^2{}_Y = \left(\frac{S_X^2}{S_X^2} \right)^{\Psi} \exp\left(\frac{S_X^2 - S_X^2}{S_X^2 + S_X^2} \right)$$

$$\frac{d}{ds_y^2} = 1$$

$$\frac{d}{ds_x^2} = U \frac{dv}{ds_x^2} + V \frac{du}{ds_y^2}$$

Assume $u = s_y^2 \left(S_X^2 \left(s_x^2 \right)^{-1} \right)^{\Psi}$ and $V = \exp\left(\frac{S_X^2 - s_x^2}{S_X^2 + s_x^2} \right)$

$$\frac{du}{ds_x^2} = -\Psi \frac{S_Y^2}{S_X^2} \text{ and } \frac{dv}{ds_x^2} = -\frac{1}{2S_X^2}$$

and

$$\frac{d}{ds^2{}_x} = \left[s_y^2 \left(\frac{S_X^2}{s_x^2} \right)^{\Psi} \times \left(\frac{-1}{2S_X^2} \right) \right] + \left[\exp\left(\frac{S_X^2 - s_x^2}{S_X^2 + s_x^2} \right) \times \left(\frac{-\Psi S_Y^2}{S_X^2} \right) \right] \tag{51}$$

therefore;

$$\frac{d}{ds^2{}_x} = \frac{-S^2{}_Y}{S^2{}_X} \left(\Psi + \frac{1}{2} \right)$$

Now, (5) becomes;

$$MSE\left(t_1^{OM}\right) = \left[1 \quad \frac{-S^2{}_Y}{S^2{}_X} \left(\Psi + \frac{1}{2} \right) \right] \begin{pmatrix} \mathrm{var}(s^2{}_y) & \mathrm{cov}(s^2{}_x s^2) \\ \mathrm{cov}(s^2{}_x s^2{}_y) & \mathrm{var}(s^2{}_x) \end{pmatrix} \begin{pmatrix} 1 \\ \frac{-S^2{}_Y}{S^2{}_X} \left(\Psi + \frac{1}{2} \right) \end{pmatrix}$$

$$= \left[\left(\mathrm{var}(s^2{}_y) - \frac{S^2{}_Y}{S^2{}_X} (\Psi + \frac{1}{2}) \mathrm{cov}(s^2{}_x s^2{}_y) \right) \quad \left(\mathrm{cov}(s^2{}_x s^2{}_y) - \frac{S^2{}_Y}{S^2{}_X} (\Psi + \frac{1}{2}) \mathrm{var}(s^2{}_x) \right) \right] \begin{pmatrix} 1 \\ \frac{-S^2{}_Y}{S^2{}_X} (\Psi + \frac{1}{2}) \end{pmatrix}$$

$$= \mathrm{var}(s^2{}_y) - \frac{S^2{}_Y}{S^2{}_X}\left(\Psi + \frac{1}{2}\right)\mathrm{cov}(s^2{}_x s^2{}_y) - \frac{S^2{}_Y}{S^2{}_X}\left(\Psi + \frac{1}{2}\right)\mathrm{cov}(s^2{}_x s^2{}_y) + \frac{\left(S^2{}_Y\right)^2}{\left(S^2{}_X\right)^2}\left(\Psi + \frac{1}{2}\right)^2 \mathrm{var}(s^2{}_x) \tag{6}$$

Where $\mathrm{var}(s_y^2) = \left(\frac{1}{n} - \frac{1}{N}\right) S_Y^4 \mu_{40}$, $\mathrm{var}(s_x^2) = \left(\frac{1}{n} - \frac{1}{N}\right) S_X^4 \mu_{04}$, and

$\mathrm{cov}(s_x^2\, s_y^2) = \left(\frac{1}{n} - \frac{1}{N}\right) S_X^2 S_Y^2 \mu_{22}$

Thus, (6) becomes;

$$MSE(t_1^{OM}) = \left(\frac{1}{n} - \frac{1}{N}\right) S^4{}_Y \left[\mu_{40} + \left(\Psi + \frac{1}{2}\right)^2 \mu_{04} - 2\left(\Psi + \frac{1}{2}\right)\mu_{22}\right] \tag{7}$$

To obtain the minimum value of Ψ, we differentiate (7) with respect to Ψ, and equate it to zero.

$$\frac{\partial MSE(t_1^{OM})}{\partial \Psi} = \left(\frac{1}{n} - \frac{1}{N}\right) S^4{}_Y \left\{2\left(\Psi + \frac{1}{2}\right)\mu_{04} - 2\mu_{22}\right\} = 0$$

$$\Psi = \frac{\mu_{22}}{\mu_{04}} - \frac{1}{2}$$

Therefore, when ignoring finite population correction term, (7) becomes;

$$MSE(t_1^{OM})_{\min} = \frac{S_y^4}{n}\left\{\mu_{40} - \frac{\mu_{22}^2}{\mu_{04}}\right\} \tag{8}$$

2.2 Properties of the Proposed Estimator t_2^{OM}

Following the procedures of obtaining $MSE(t_1^{OM})$, then, $MSE(t_2^{OM})$ can be obtained as follows;

$$\frac{d}{ds^2{}_y} \mid S_X^2 S_Y^2 = \left[\frac{(1+\Delta)S_X^2 + (1-\Delta)S_X^2}{(1-\Delta)S_X^2 + (1+\Delta)S_X^2}\right]\exp\left(\frac{S_X^2 - S_X^2}{S_X^2 + S_X^2}\right) = 1$$ and

$$\frac{d}{ds^2{}_x} = U\frac{dv}{ds^2{}_x} + V\frac{du}{ds^2{}_x}$$

If $U = \frac{s^2{}_y\left(1+\Delta\right)S^2{}_X + s^2{}_y\left(1-\Delta\right)s^2{}_x}{\left(1-\Delta\right)S^2{}_X + \left(1+\Delta\right)s^2{}_x}$, $V = \exp\left(\frac{S^2{}_X - s^2{}_x}{S^2{}_X + s^2{}_x}\right)$, then,

$\frac{du}{ds^2{}_x} = \frac{-\Delta S^2{}_Y}{S^2{}_X}$, $\frac{dv}{ds^2{}_x} = \frac{-1}{2S^2{}_X}$ and $\frac{d}{ds^2{}_x} = \frac{-S^2{}_Y}{S^2{}_X}\left(\Delta + \frac{1}{2}\right)$

Therefore;

$$MSE\left(t_2^{OM}\right) = \begin{bmatrix} 1 & \frac{-S^2{}_Y}{S^2{}_X}\left(\Delta+\frac{1}{2}\right) \end{bmatrix} \begin{pmatrix} \mathrm{var}(s_y^2) & \mathrm{cov}(s_x^2\, s_y^2) \\ \mathrm{cov}(s_x^2\, s_y^2) & \mathrm{var}(s_x^2) \end{pmatrix} \begin{pmatrix} 1 \\ \frac{-S_Y^2}{S_X^2}\left(\Delta+\frac{1}{2}\right) \end{pmatrix}$$

And

$$MSE(t_2^{OM}) = \left(\frac{1}{n} - \frac{1}{N}\right)S^4{}_Y\left[\mu_{40} + \left(\Delta+\frac{1}{2}\right)^2\mu_{04} - 2\left(\Delta+\frac{1}{2}\right)\mu_{22}\right] \tag{9}$$

To obtain minimum value of Δ, we differentiate $MSE(t_2^{OM})$ with respect to Δ, and equate it to zero.

$$\frac{\partial MSE(t_2^{OM})}{\partial\Delta} = \left(\frac{1}{n} - \frac{1}{N}\right)S^4{}_Y\left\{2\left(\Delta+\frac{1}{2}\right)\mu_{04} - 2\mu_{22}\right\} = 0$$

$$\Delta = \frac{\mu_{22}}{\mu_{04}} - \frac{1}{2}$$

Now, when ignoring finite population correction term, (9) becomes;

$$MSE(t_2^{OM})_{\min} = \frac{S_y^4}{n}\left\{\mu_{40} - \frac{\mu_{22}^2}{\mu_{04}}\right\} \tag{10}$$

2.3 Properties of the Proposed Estimator t_3^{OM}

$$\frac{d}{ds_y^2} / S_X^2 S_X^2 = \left[\frac{\cancel{S}_X^2}{\Theta\cancel{S}_X^2 + \cancel{S}_X^2 - \Theta\cancel{S}_X^2}\right]\exp\left(\frac{S_X^2 - S_X^2}{S_X^2 + S_X^2}\right) = 1,$$

$$\frac{d}{ds^2{}_x} / S^2{}_X S^2{}_Y = \frac{udv}{ds^2{}_x} + \frac{vdu}{ds^2{}_x}$$

Where $u = \frac{s^2{}_y S^2{}_X}{\alpha s^2{}_x + (1-\alpha)S^2{}_X}$, $v = \exp\left(\frac{S^2{}_X - s^2{}_x}{S^2{}_X + s^2{}_x}\right)$

$$\frac{du}{ds^2{}_x} / S^2{}_X S^2{}_Y = \frac{\frac{v_1 du_1}{ds_x^2} - \frac{u_1 dv_1}{ds_x^2}}{(v_1)^2}$$

$$= \frac{\left[\Theta s^2{}_x + (1-\Theta) S^2{}_X\right] \times 0 - \left[\left(S^2{}_X s^2{}_y\right)\Theta\right]}{\left[\Theta s^2{}_x + (1-\Theta) S^2{}_X\right]^2} = -\frac{\Theta S_Y^2}{S_X^2}$$

$$\frac{dv}{ds^2{}_x} = \exp\left(\frac{S^2{}_X - s^2{}_x}{S^2{}_X + s^2{}_x}\right)\left[\frac{\frac{v_2 du_2}{ds_x^2} - u_2 \frac{dv_2}{ds_x^2}}{(v_2)^2}\right] = -\frac{1}{2S_X^2}$$

and,

$$\frac{d}{ds^2{}_x} = \frac{-S^2{}_Y}{S^2{}_X} + \left[\exp(0) \times \frac{-\Theta S^2{}_Y}{S^2{}_X}\right] = \frac{-S^2{}_Y}{2S^2{}_X} - \frac{\alpha S^2{}_Y}{S^2{}_X} = -\frac{S_Y^2}{S_X^2}\left(\Theta + \frac{1}{2}\right)$$

Therefore,

$$MSE\left(t_3^{OM}\right) = \begin{bmatrix} 1 & \frac{-S_Y^2}{S_X^2}\left(\Theta + \frac{1}{2}\right) \end{bmatrix} \begin{pmatrix} \text{var}(s_y^2) & \text{cov}(s_x^2 s_y^2) \\ \text{cov}(s_x^2 s_y^2) & \text{var}(s_x^2) \end{pmatrix} \begin{pmatrix} 1 \\ \frac{-S_Y^2}{S_X^2}\left(\Theta + \frac{1}{2}\right) \end{pmatrix}$$

$$MSE(t_3^{OM})_{\min} = \frac{S_y^4}{n}\left\{\mu_{40} - \frac{\mu_{22}^2}{\mu_{04}}\right\} \quad (11)$$

Where $\Theta = \frac{\mu_{22}}{\mu_{04}} - \frac{1}{2}$

3. Empirical Study

To have reasonable comparison of the performances of the proposed estimators over the existing estimators numerically, six natural population data sets were considered and their descriptions are given below.

Table 1: MSE and PRE of different estimators using Population 1

ESTIMATORS	**MSE**	**PRE**
Sample variance $t_{s_y^2}$	5393.75	100.00
t_{Isaki}	2952.368	182.6923
$t_{Isaki}^{\text{Re}g}$	1992.479	270.7055
t_{BT}	2188.727	246.4332
t_{sh}	1958.12	275.4555
t_{SG}	1915.383	281.6017
t_{YK}	1992.479	270.7055
t_{SM}	218069748	0.002473406
$t_{sh_1}^{\tau}, \tau = AM, GM, HM$	1309.971	411.7456
$t_{sh_2}^{\tau}, \tau = AM, GM, HM$	3028.91	178.0756
$t_{sh_3}^{\tau}, \tau = AM, GM, HM$	251.1287	2147.803
$t_{sh_4}^{\tau}, \tau = AM, GM, HM$	1309.971	411.7456
t_{Swain}	1992.479	270.7055
t_{YS}	1911.195	282.2187
t_{SH}^{CR}	4858.349	111.0202
$t_{SH}^{C\text{Re}}$	147.6184	3653.846
$t_{SH}^{CR\text{Re}}$	151.1933	3567.452
t_{Muneer}	1631.371	330.6269
$t_j^{OM}, j = 1, 2, 3$	100.0498	5391.066

Table 1 shows the MSEs and PREs of the proposed estimators and the existing estimators using population 1. The proposed estimators have least MSEs compared to other competing estimators. Thus, this implies that the estimates of the proposed

estimators are on average closer to the true estimate than that of other competing estimators with gain in efficiency over other estimators.

Table 2: MSE and PRE of different estimators using Population 2

ESTIMATORS	MSE	PRE
Sample variance $t_{s_y^2}$	97.75015	100.00
t_{Isaki}	29.18823	334.8958
$t_{Isaki}^{\mathrm{Re}g}$	20.33966	480.589
t_{BT}	28.77017	339.7622
t_{sh}	20.07112	487.019
t_{SG}	19.41761	503.4099
t_{YK}	20.33966	480.589
t_{SM}	30307.5	0.3225279
$t_{sh_1}^{\tau}, \tau = AM, GM, HM$	25.09178	389.5705
$t_{sh_2}^{\tau}, \tau = AM, GM, HM$	57.18131	170.9477
$t_{sh_3}^{\tau}, \tau = AM, GM, HM$	5.33638	1831.769
$t_{sh_4}^{\tau}, \tau = AM, GM, HM$	25.09178	389.5705
t_{Swain}	20.33966	480.589
t_{YS}	19.35079	505.1482
t_{SH}^{CR}	91.57154	106.7473
$t_{SH}^{C\mathrm{Re}}$	2.084873	4688.542
$t_{SH}^{CR\mathrm{Re}}$	2.148638	4549.4
t_{Muneer}	14.5463	671.9931
$t_j^{OM}, j = 1, 2, 3$	1.450227	6740.334

Table 2 shows the MSEs and PREs of the proposed estimators and the existing estimators using population 2. The proposed estimators have least MSEs and highest PREs compared to other competing estimators. Thus, these results are in conformity with that of population 1 in Table 1.

Table 3: MSE and PRE of different estimators using Population 3

ESTIMATORS	MSE	PRE
Sample variance $t_{s_y^2}$	38482180	100.00
t_{Isaki}	21898276	175.7316
$t_{Isaki}^{\mathrm{Re}g}$	21871385	175.9476
t_{BT}	25695997	149.7594
t_{sh}	16644713	231.1976
t_{SG}	15515541	248.0235
t_{YK}	21871385	175.9476
t_{SM}	1.30173e+15	2.956234e-06
$t_{sh_1}^{\tau}, \tau = AM, GM, HM$	2070811	1858.314
$t_{sh_2}^{\tau}, \tau = AM, GM, HM$	2053290	1874.172
$t_{sh_3}^{\tau}, \tau = AM, GM, HM$	458009.8	8402.043
$t_{sh_4}^{\tau}, \tau = AM, GM, HM$	2070811	1858.314
t_{Swain}	21871385	175.9476
t_{YS}	14417460	266.9137
t_{SH}^{CR}	709569.6	5423.313
$t_{SH}^{C\mathrm{Re}}$	547456.9	7029.262
$t_{SH}^{CR\mathrm{Re}}$	547691.6	7026.25
t_{Muneer}	13806309	278.7289
$t_j^{OM}, j = 1, 2, 3$	546782.2	7037.935

Table 3 shows the MSEs and PREs of the proposed estimators and the existing estimators using population 3. The proposed estimators have least MSEs and highest PREs compared to other competing estimators. Thus, these results are in conformity with that of Table 1 and 2.

Table 4: MSE and PRE of different estimators using Population 4

ESTIMATORS	MSE	PRE
Sample variance $t_{s_y^2}$	476.3475	100.00
t_{Isaki}	248.5163	191.6766
$t_{Isaki}^{\mathrm{Re}g}$	150.0079	317.5483
t_{BT}	166.2932	286.4505
t_{sh}	142.7608	333.6683
t_{SG}	130.1876	365.8931
t_{YK}	150.0079	317.5483
t_{SM}	389790.1	0.1222062
$t_{sh_1}^{\tau}, \tau = AM, GM, HM$	114.4918	416.0537
$t_{sh_2}^{\tau}, \tau = AM, GM, HM$	277.9814	171.3595
$t_{sh_3}^{\tau}, \tau = AM, GM, HM$	52.63611	904.9825
$t_{sh_4}^{\tau}, \tau = AM, GM, HM$	114.4918	416.0537
t_{Swain}	150.0079	317.5483
t_{YS}	125.5534	379.3985
t_{SH}^{CR}	273.0429	174.4589
$t_{SH}^{C\mathrm{Re}}$	49.70326	958.3829
$t_{SH}^{CR\mathrm{Re}}$	56.84327	838.0016
t_{Muneer}	51.40574	926.6427
$t_j^{OM}, j = 1, 2, 3$	29.95057	1590.445

Table 4 shows the MSEs and PREs of the proposed estimators and the existing estimators using population 4. The proposed estimators have least MSEs and highest PREs compared to other competing estimators. Thus, these results are in conformity with that of Table 1, 2 and 3.

Table 5: MSE and PRE of different estimators using Population 5

ESTIMATORS	**MSE**	**PRE**
Sample variance $t_{s_y^2}$	14396.5	100.00
t_{Isaki}	4863.133	296.0336
$t_{Isaki}^{\mathrm{Re}g}$	4317.988	333.4077
t_{BT}	5800.705	248.1855
t_{sh}	3502.563	411.0277
t_{SG}	2619.15	549.6632
t_{YK}	4317.988	333.4077
t_{SM}	67445938	0.02134525
$t_{sh_1}^{\tau}, \tau = AM, GM, HM$	581.9252	24739.44
$t_{sh_2}^{\tau}, \tau = AM, GM, HM$	645.7979	222.9259
$t_{sh_3}^{\tau}, \tau = AM, GM, HM$	160.7713	895.4647
$t_{sh_4}^{\tau}, \tau = AM, GM, HM$	581.9252	2473.944
t_{Swain}	4317.988	333.4077
t_{YS}	903.9956	1592.541
t_{SH}^{CR}	370.7629	3882.941
$t_{SH}^{C\mathrm{Re}}$	243.1566	5920.671
$t_{SH}^{CR\mathrm{Re}}$	372.5457	3864.36
t_{Muneer}	318.978	4513.322
$t_j^{OM}, j = 1, 2, 3$	215.8624	6669.297

Table 5 shows the MSEs and PREs of the proposed estimators and the existing estimators using population 5. The proposed estimators have least MSEs and highest PREs compared to other competing estimators. Thus, these results are in conformity with that of Table 1, 2, 3 and 4.

Table 6: MSE and PRE of different estimators using Population 6

ESTIMATORS	**MSE**	**PRE**
Sample variance $t_{s_y^2}$	0.3401182	100.00
t_{Isaki}	0.4220369	80.58968
$t_{Isaki}^{\mathrm{Re}g}$	0.3100965	109.6814
t_{BT}	0.3167869	107.365
t_{sh}	0.2925965	116.2414
t_{SG}	0.2887802	117.7775
t_{YK}	0.3100965	109.6814
t_{SM}	1.475151	23.0565
$t_{sh_1}^{\tau}, \tau = AM, GM, HM$	0.05154712	659.82
$t_{sh_2}^{\tau}, \tau = AM, GM, HM$	0.1370542	248.1633
$t_{sh_3}^{\tau}, \tau = AM, GM, HM$	0.1541775	220.6018
$t_{sh_4}^{\tau}, \tau = AM, GM, HM$	0.05154712	659.82
t_{Swain}	0.3100965	109.6814
t_{YS}	0.2880048	118.0946
t_{SH}^{CR}	0.2192867	155.102
$t_{SH}^{C\mathrm{Re}}$	0.02221247	1531.204
$t_{SH}^{CR\mathrm{Re}}$	0.02251949	1510.328
t_{Muneer}	0.2723603	124.878
$t_j^{OM}, j = 1, 2, 3$	0.01632964	2082.827

Table 6 shows the MSEs and PREs of the proposed estimators and the existing estimators using population 6. The proposed estimators have least MSEs and highest PREs compared to other competing estimators. Thus, these results are in conformity with that of Table 1, 2, 3, 4 and 5.

4. Simulation Study

In this section, simulation study was performed to investigate the superiority of the suggested estimators over other competing estimators in the study. We considered a bivariate random population of size N=1000 generated using R programming language. A sample of size n=80 was drawn 10,000 times by SRSWOR method from it. The values obtained for the MSEs and the PREs of all the estimators are summarized in Table 7.

Table 6: MSE and PRE of different estimators using simulation study

Estimators	**MSE**	**PRE**
Sample variance $t_{s_y^2}$	1.57936e+22	100
t_{Isaki}	5.850574e+14	2699495370
$t_{Isaki}^{\mathrm{Re}g}$	1.579721e+22	99.97712
t_{BT}	2.139069e+21	738.3398
t_{sh}	14113834	1.119015e+17
t_{SG}	3.034931e+16	52039387
t_{YK}	1.57936e+22	100
t_{SM}	2.631981e+32	6.000651e-09
$t_{sh_1}^{AM}$	3.949912e+21	399.8468
$t_{sh_1}^{GM}$	3.02504e+18	522095.5
$t_{sh_1}^{HM}$	6.31502e+22	25.00958
$t_{sh_2}^{AM}$	7.389349e+21	213.7346
$t_{sh_2}^{GM}$	5.812365e+21	271.7241
$t_{sh_2}^{HM}$	3.375638e+22	46.78699
$t_{sh_3}^{AM}$	5.35324e+20	2950.288
$t_{sh_3}^{GM}$	1.11328e+18	1418655
$t_{sh_3}^{HM}$	2.33996e+15	674951693
t_{Swain}	2.010337e+17	7856193
t_{YS}	1.255689e+33	1.257763e-09
t_{SH}^{CR}	16542098	9.547517e+16
$t_{SH}^{C\,\mathrm{Re}}$	2.897133e+20	5451.457

$t_{SH}^{CR\mathrm{Re}}$	5.549104e+16	28461529
t_{Muneer}	8.43866e+27	0.0001871576
t_1^{OM}	475894.2	3.318721e+18
t_2^{OM}	2.139069e+21	738.3398
t_3^{OM}	18295797626	8.632363e+13

Table 7 shows the MSEs and PREs of the proposed estimators and the existing estimators using simulated data. The proposed estimators with the exception of t_2^{OM} and t_3^{OM} , performed below Singh *et al.* (2018) and Singh *et al*. (1988) estimators, have least MSEs and highest PREs compared to other competing estimators. Thus, these results are in conformity with that of Table 1, 2, 3, 4, 5 and 6.

5. Conclusion and Recommendation

From the results Tables 1, 2, 3, 4, 5, and 6, it is observed that the proposed estimators are more efficient than all the competing estimators considered in the study making them more applicable in estimating population variances. Also, results of simulation study summarized in Table 7 revealed the superiority of proposed estimator tom1 over other estimators. However, the proposed estimators t_2^{OM} and t_3^{OM} performed below the estimator of Singh *et al.* (2018) and Singh *et al*. (1988) under simulation study and generally the efficiency of the proposed estimators improved more the competing ones.

References

1. Bahl S, Tuteja RK (1991). Ratio and product type exponential estimators. *Inf Optim Sci. 12(1),* 159–164.
2. Biradar, R. S, Singh, H. P. (1998). *Predictive estimation of finite population variance.* Calcutta Statistical Association Bulletin, *48*, 229-235.
3. Das, A. K., and T. P. Tripathi. (1978). Use of auxiliary information in estimating the finite population variance. *Sankhya 402*, 139–48.
4. Gujarati DM (2004). Basic econometrics. New York: The McGraw-Hill Companies.
5. Grover LK (2010). A correction note on improvement in variance estimation using auxiliary information. *Commun Stat-Theory Methods.43,* 101–105.
6. Grover, L. K. (2007). A wide and efficient class of estimators of population variance under sub-sampling scheme. *Model Assisted Statistics and Applications*, *2(1),* 41-51.

7. Ishaq, O. O., Audu, A., Ibrahim, A., Abdulkadir, H. S., & Kabiru, T. (2020). On the linear combination of sample variance, ratio and product estimators of finite population variance under two stage sampling. *Journal of Pure and Applied Sciences, 20(3),* 307-315.
8. Kadilar C, Cingi H (2005). A new estimator using two auxiliary variables. *J Appl Math Compu;162*, 901–908.
9. Kadilar, C., and H. Cingi. (2006). Ratio estimators for the population variance in simple and stratified sampling. *Applied Mathematics and Computation 173,* 1047–59.
10. Kadilar, C and Cingi, H. (2007). *Improvement in Variance Estimation in Simple Random Sampling.* Communications in Statistics-Theory and Methods*, 36(11),* 2075-2081.
11. Isaki CT (1983). Variance estimation using auxiliary information. *J Am Stat Assoc.78,* 117–123.
12. Mukherjee C, White H, Whyte M (1998). Econometrics and data analysis for developing countries. Routledge;London.
13. Murthy MN (1967). Sampling theory and methods. Calcutta: *Statistical Public Society*.
14. Muneer, S., Khalil, A., Shabbir, J., & Narjis, G. (2018). A new improved ratio-product type exponential estimator of finite population variance using auxiliary information. *Journal of Statistical Computation and Simulation*, *88(16),* 3179-3192.
15. Muili, J. O., Zakari, Y., & Audu, A. (2019). Modified Class of Estimator of Finite Population Variance. *FUDMA JOURNAL OF SCIENCES-ISSN: 2616-1370*, 3(4), 67-78.
16. Swain AKPC (2015). Generalized estimator of finite population variance. *J Stat Theory Appl.14(1),* 45–51.
17. Sumramani J, Kumarapandiyan G (2015). Generalized modified ratio type estimator for estimation of population variance. *Sri-Lankan J Appl Stat.16(1),* 69–90.
18. Shabbir J, Gupta S (2007). On improvement in variance estimation using auxiliary information.*Commun Stat-Theory Methods.36*, 2177–2185.
19. Singh R, Malik S (2014). Improved estimation of population variance using information on auxiliary attribute in simple random sampling. *Appl Math Comput. 235*, 43–49.
20. Singh HP, Solanki RS (2013). Improved estimation of finite population variance using auxiliary information*. Commun Stat-Theory Methods*, *42,* 2718–2730.
21. Singh, H. P., Singh, A. K., & Solanki, R. S. (2014). Estimation of finite population variance using auxiliary information in sample surveys. *Statistica*, *74(1),* 99-116.

22. Singh, H. P., Pal, S. K., & Yadav, A. (2018). A study on the chain ratio-ratio-type exponential estimator for finite population variance. *Communications in Statistics-Theory and Methods, 47(6),* 1442-1458.
23. Singh HP, Singh R (2001). Improved ratio type estimator for variance using auxiliary information. *J Ind Soc Agri Stat. 54(3)*, 276–287.
24. Singh, H. P., Upadhyaya, L. N., & Namjoshi, U. D. (1988). Estimation of finite population variance. *Current Science*, 1331-1334.
25. Singh, H. P., Chandra, P., & Singh, S. (2003). Variance estimation using multiauxiliary information for random non-response in survey sampling. *Statistica*, *63(1),* 23-40.
26. Singh, H. P., & Vishwakarma, G. K. (2008). Some families of estimators of variance of stratified random sample mean using auxiliary information. *Journal of Statistical Theory and Practice*, *2(1),* 21-43.
27. Singh, H. P., Singh, A. K., & Solanki, R. S. (2014). Estimation of finite population variance using auxiliary information in sample surveys. *Statistica*, *74(1),* 99-116.
28. Singh, H. P., & Solanki, R. S. (2013). Improved estimation of finite population variance using auxiliary information. *Communications in Statistics-Theory and Methods*, *42(15),* 2718-2730.
29. Singh, H. P., & Solanki, R. S. (2013a). A new procedure for variance estimation in simple random sampling using auxiliary information. *Statistical Papers*, *54(2),* 479-497.
30. Singh, H. P., & Solanki, R. S. (2013b). Improved estimation of finite population variance using auxiliary information. *Communications in Statistics-Theory and Methods*, *42(15),* 2718-2730.
31. Solanki, R. S. and Singh, H. P. (2013). An improved class of estimators for the population variance. *Model Assisted Statistics and Applications, 8(3),* 229-238.
32. Srivastava, S. K., Jhajj, H. S. (1980). *A class of estimators using auxiliary information for estimating finite population variance.* Sankhya, C 42: 87-96.
33. Upadhyaya L. N., Singh H. P., Singh S. (2004). A class of estimators for estimating the variance of the ratio estimator. *J Jap Stat Soc. 34(1),* 47–63.
34. Upadhyaya L. N., Singh H. P. (1999). An estimator of population variance that utilizes the kurtosis of an auxiliary variable in sample surveys. *Vikram Mathemat J*ournal *19,* 14–17.
35. Yadav S. K. and Kadilar, C. (2014). A two-parameter variance estimator using auxiliary information. *Appl Math Comput. 226,* 117–122.
36. Yaqub M, Shabbir J (2016). An improved class of estimators for finite population variance. *Hacettepe J Math Stat. 45(5),* 1641–1660.