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Statistical Inference for Weibull Distribution Based on a Modified Progressive Type-II Censoring Scheme

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ABSTRACT

In this paper, a modified progressive Type-II censoring scheme is introduced. Relationships between the modified progressive Type-II censoring scheme and a randomized Type-II censoring scheme are discussed. The maximum likelihood estimators (MLEs) of the parameters of Weibull distribution are derived and EM-algorithm is used to obtain the estimates as well as the asymptotic variance-covariance matrix. Monte Carlo simulation is used to evaluate the performance of the MLEs in terms of biases, mean square errors and some commonly used optimal criteria in experimental design. Finally, a numerical example is provided to illustrate the methodology presented here.

Keywords: EM-algorithm, Maximum likelihood estimates, Optimality criteria, Progressive censoring, Asymptotic variances, Missing information

1. Introduction

Because of the increase in global competition and the customer expectations for reliable products, manufacturers need to improve the product quality and reliability and to provide a longer warranty period in order to stay competitive in the market. In reliability engineering, knowledge about failure time distribution of the products are valuable in determine optimal warranty policy (Wu and Huang, 2012) and product design improvement. To obtain knowledge about product lifetime distribution, life-testing experiments are run at different product development and testing stages before the product can be put on the market.

When the product are extremely reliable or the cost of a failing a product is high, a censoring scheme is always imposed in the life-testing experiment in order to save time and cost of the experiment. One of the commonly used censoring schemes is the Type-II right censoring scheme whereas the life-testing experiment is terminated as soon as the m-th (where m is pre-fixed) failure is observed. Then, only the first m failures out of n units under test will be observed. The data obtained from such a restrained life-test will be referred to as a *Type-II censored sample*. A generalization of the Type-II censoring scheme, called progressive Type-II censoring scheme, has been proposed in the literature. Under the progressive censoring scheme, n independent units are placed on a life test and m failures are going to be observed. Immediately following the first failure time (say, $X^{\mathbf{R}}_{1:m:n}$), R_1 of the surviving units are randomly selected and removed from the experiment. Then, immediately following the second failure time (say, $X^{\mathbf{R}}_{2:m:n}$), R_2 of the surviving units are removed, and so on. This experiment terminates at the time when the m-th failure is observed and the remaining $n - m - R_1 - \cdots - R_{m-1}$ surviving units are censored. Here, $X^{\mathbf{R}}_{1:m:n} < X^{\mathbf{R}}_{2:m:n} < \cdots < X^{\mathbf{R}}_{m:m:n}$ describe the progressively censored failure times where $\mathbf{R} = (R_1, \ldots, R_m)$ is the progressive censoring scheme. Two books dedicated to progressive censoring were written by Balakrishnan and Aggarwala (2000) and Balakrishnan and Cramer (2014) and a comprehensive review of the subject was provided in a discussion paper by Balakrishnan (2007). Inference and generalizations based on progressive censoring scheme were studied by Cohen (1963), Mann (1971), Gibbons and Vance (1983), Viveros and Balakrishnan (1994), Yuen and Tse (1996), Balasooriya et al. (2000), Wu (2002), Ng et al. (2002, 2004), Soliman (2005), Asgharzadeh (2006), Wu et al. (2006a, 2006b, 2007), Kuş and Kaya (2007), Kuş and Wu (2008), Wu and Kuş (2009) and Saraçoğlu et al. (2012).

Recently, some modifications of the progressive Type-II censoring scheme have been studied. For example, Kundu and Joarder (2006) proposed a progressive hybrid censoring scheme, Wu and Kuş (2009) introduced a progressive first failure censoring scheme, Ng et al. (2009) proposed an adaptive progressive censoring scheme and Bairamov and Parsi (2011) proposed a flexible progressive censoring scheme. Extensive research has been done related to these modifications of the progressive censoring, see, for example, Lin et al. (2009), Cramer and Iliopoulos (2010), Soliman et al. (2012a, b), Wu and Huang (2012) and Lio and Tsai (2012).

In this paper, a simple modification of the progressive censoring, which can be viewed as a randomized Type-II censoring scheme, is developed. The main objective of this paper is to compare the proposed modification with the progressive Type-II censoring scheme. For this purpose, the performance of the estimators for the Weibull parameters based on data obtained from the modified censoring scheme as well as the progressive censoring scheme are examined. The rest of this paper is organized as follows. In Section 2, we describe the formulation of the modified censoring scheme (MCS) and the relationship between the MCS and a randomized Type-II censoring scheme. In Section 3, we consider Weibull lifetimes and obtain the maximum likelihood estimates (MLEs) of the parameters based on log-lifetimes under the MCS by EM algorithm. Fisher information matrix of the MLEs is also obtained by using missing information principle. Section 4 presents the results of a Monte Carlo simulation study for assessing the performance of the MLEs under MCS. A numerical example is presented in Section 5 to illustrate the methodology presented in this paper. In Section 6, some concluding remarks are provided.

2. Modified Censoring Scheme

In a progressively Type-II censored experiment, n independent units are placed on a life test simultaneously. Suppose that the experimenter decided to observe $m(\leq n)$ failures and the progressive censoring scheme $\mathbf{R} = (R_1, R_2, \ldots, R_m)$ are prefixed, then R_i units are randomly removed from the test as soon as the i-th failure is observed $(i = 1, 2, \ldots, m)$. Denote the i-th observed failure time from this scheme as $X^{\mathbf{R}}_{i:m:n}$, the exact failure times of the R_i censored units cannot be observed beyond $X^{\mathbf{R}}_{i:m:n}$, $i = 1, 2, \ldots, m$. In the modified censoring scheme proposed here, we allow the R_i $(i = 1, 2, \ldots, m-1)$ units to be monitored until to the end of the progressively censored experiment, i.e., until $X^{\mathbf{R}}_{m:m:n}$ is observed. Hence, the exact failure times of some of the $\sum_{i=1}^{m-1} R_i$ censored units in the progressively censored experiment can be recorded in addition to the progressively censored order statistics. We can see that the number of total observed failures before $X^{\mathbf{R}}_{m:m:n}$, say $m + K$, is a random variable, where K is a discrete random variable with support $\{0, 1, 2, \ldots, n-m\}$.

This modified censoring scheme (MCS) can be viewed as a randomized Type-II censoring scheme in which the first $m + K$ failures are observed among the n units with K being a discrete random variable with support $\{0, 1, 2, \ldots, n-m\}$

and probability mass function (p.m.f.)

$$\Pr(K = k) = p_k, k = 0, 1, \ldots, n - m,$$

where $0 \le p_k \le 1$ and $\sum_{k=0}^{n-m} p_k = 1$. Therefore, the observed ordered failures from the MCS can be denoted as $X_{1:n} < X_{2:n} < \cdots < X_{m+K:n}$, where $X_{i:n}$ is the i-th ordinary order statistics from a sample of size n. Note that $X_{m+K:n} \stackrel{d}{=} X^{\mathbf{R}}_{m:m:n}$.

Here, the p.m.f. of K is determined by the progressive censoring scheme $\mathbf{R}$. For instance, the progressive censoring scheme $\mathbf{R} = (0, 0, \ldots, 0, n - m)$ (a conventional Type-II censoring scheme) gives the p.m.f. of K as

$$\Pr(K = 0) = 1 \text{ and } \Pr(K = k) = 0, k = 1, 2, \ldots, n - m;$$

and the progressive censoring scheme $\mathbf{R} = (0, 0, \ldots, 0, 1, n - m - 1)$ gives the p.m.f. of K as

$$\begin{aligned} \Pr(K = 0) &= \frac{n - m}{n - m + 1}, \Pr(K = 1) = \frac{1}{n - m + 1}, \\ \Pr(K = k) &= 0, k = 2, 3, \ldots, n - m. \end{aligned}$$

This result can be seen by observing that the event $\{K = 1\}$ is equivalent to that the $R_{m-1} = 1$ item being removed at the the time of the $(m-1)$-th failure is the item with the shortest lifetime among the $(n - m + 1)$ surviving items. In general, the p.m.f. of K can be expressed as (Ng, et al., 2014)

$$\begin{aligned} \Pr(K = k) &= \Pr(X^{\mathbf{R}}_{m:m:n} = X_{m+k:n}) \\ &= \sum_{a_2=a_1+1}^{b_2} \sum_{a_3=a_2+1}^{b_3} \cdots \sum_{a_{m-1}=a_{m-2}+1}^{b_{m-1}} \\ &\qquad \left\{ \prod_{i=1}^{m-1} \frac{\binom{n - a_{i+1}}{\sum_{j=1}^{i} R_j - a_{i+1} + i + 1}}{\binom{n - a_i}{\sum_{j=1}^{i} R_j - a_i + i}} \right\}, \end{aligned}$$

for $K = 0, 1, \ldots, n - m$, where $a_1 = 1$, $a_m = m + k$, $b_i = \min\left(a_{i-1} + \sum_{j=1}^{i} R_j + 1, n\right)$.

Since the number of observed failures under the MCS must be greater than or equal to m, one should expect to obtain the same or more information about the lifetime distribution compare to a progressive Type-II censoring scheme. The MCS guarantees at least m observed failures are available for statistical inference and the total time required for the experiment is determined by a chance process.

Suppose that the lifetimes of the units are independent and identically distributed (i.i.d.) with probability density function (p.d.f.) $f(x;\boldsymbol{\theta})$ and cumulative distribution function (c.d.f.) $F(x;\boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is the parameter vector. Then, given $K = k$, the conditional likelihood function based on a sample obtained from MCS can be written as

$$L\left(\boldsymbol{\theta};\boldsymbol{x}\right) \propto \left(1 - F\left(x_{m+k:n};\boldsymbol{\theta}\right)\right)^{n-(m+k)} \prod_{i=1}^{m+k} f\left(x_{i:n};\boldsymbol{\theta}\right),\ \boldsymbol{\theta} \in \boldsymbol{\Theta}, \quad (2.1)$$

where $x_{i:n}$ is the observed value of $X_{i:n}$ and $\boldsymbol{\Theta}$ is the parameter space.

3. Statistical Estimation with Weibull Lifetimes

Suppose n independent units are placed on a life-test with the corresponding lifetimes X_1, X_2, …, X_n being identically distributed. We assume that X_i, $i = 1, 2, \ldots, n$ are i.i.d. Weibull distributed with p.d.f. and c.d.f.

$$\begin{aligned} g\left(x;\lambda,\beta\right) &= \beta\lambda^{-\beta}x^{\beta-1}\exp\left\{-\left(\frac{x}{\lambda}\right)^{\beta}\right\} \\ \text{and } G\left(x;\lambda,\beta\right) &= 1-\exp\left\{-\left(\frac{x}{\lambda}\right)^{\beta}\right\}, \qquad x>0,\ \lambda>0,\ \beta>0, \end{aligned}$$

respectively, where $\lambda > 0$ is the scale parameter and $\beta > 0$ is the shape parameter. For $0 < \beta < 1$, the Weibull distribution has a decreasing hazard function. For $\beta > 1$, the Weibull distribution has an increasing hazard function and for $\beta = 1$, the Weibull distribution reduces to an exponential distribution which has a constant hazard rate. Here, we can consider the log-transformed lifetime $Y = \ln X$ which has an extreme-value distribution with location parameter $\mu = \ln\lambda$ and scale parameter $\sigma = \beta^{-1}$. Specifically, the p.d.f. and c.d.f. of Y are

$$f\left(y;\boldsymbol{\theta}\right) = \frac{1}{\sigma}\exp\left(\frac{y-\mu}{\sigma}\right)\exp\left\{-\exp\left(\frac{y-\mu}{\sigma}\right)\right\}, \quad (3.1)$$

$$\text{and } F\left(y;\boldsymbol{\theta}\right) = 1-\exp\left\{-\exp\left(\frac{y-\mu}{\sigma}\right)\right\}, \quad (3.2)$$

for $y > 0,\ \mu \in \mathbb{R},\ \sigma > 0$, respectively, where $\boldsymbol{\theta} = (\mu,\sigma)'$. Since the extreme-value distribution is a member of the location-scale family of distributions and it is easier to work with, therefore, the log-lifetimes are used in the statistical analysis (see, for example, Ng et al., 2002).

Under the progressive Type-II censoring scheme, the MLEs of the Weibull parameters λ and β (or equivalently μ and σ) and Fisher information matrix were obtained by Ng et al. (2002) using EM-algorithm and missing information principle, respectively. Since the MCS is a randomized Type-II censoring scheme, given the value of $K = k$, the conditional statistical inference can be done by using the results from conventional Type-II censoring, which has been well-developed in the literature. Therefore, we only briefly present the computational formulae for the MLEs and the Fisher information matrix as well as the steps required for EM-algorithm if it is chosen to be used to obtain the MLEs.

3.1. Maximum Likelihood Estimates

Given $K = k$, based on the ordered log-lifetimes $Y_{1:n} < Y_{2:n} < \cdots < Y_{m+k:n}$, the MLE of parameter σ can be obtained by solving the following equation with respect to σ:

$$\begin{aligned} m+k &= \left\{ -\sum_{j=1}^{m+k} \left(\frac{y_{j:n} - g(\sigma)}{\sigma} \right) + \sum_{j=1}^{m+k} \left(\frac{y_{j:n} - g(\sigma)}{\sigma} \right) e^{\frac{y_{j:n} - g(\sigma)}{\sigma}} \right. \\ &\quad \left. + (n-m-k) \left(\frac{y_{j:n} - g(\sigma)}{\sigma} \right) e^{\frac{y_{j:n} - g(\sigma)}{\sigma}} \right\}, \end{aligned}$$

where

$$\begin{aligned} & g(\sigma) \\ = \; & \sigma \log \left\{ \frac{1}{m+k} \left[\sum_{j=1}^{m+k} \exp\left(\frac{y_{j:n}}{\sigma} \right) + (n-m-k) \exp\left(\frac{y_{m+k:n}}{\sigma} \right) \right] \right\}. \end{aligned}$$

Numerical methods, such as the Newton-Raphson method, can be used here to solve the above non-linear equation. After obtaining the MLE of σ, say $\hat{\sigma}$, the MLE of μ can be obtained as $\hat{\mu} = g(\hat{\sigma})$.

Besides using direct optimization numerical methods, Expectation-Maximization (EM)-algorithm can be used as an alternative tool here to obtain the MLEs of μ and σ. One of the advantage of EM-algorithm is that the second derivatives of the log-likelihood is not required to implement the EM-iteration. It is especially useful if the complete data set is easy to analyze (see, for example, Adamidis and Loukas, 1998; Adamidis, 1999; Ng et al., 2002; Karlis, 2003;

Adamidis et al., 2005; Kuş, 2007; Tahmasbi and Rezaei, 2008; and Morais and Barreto-Souza, 2011). The analysis with the data from MCS can be viewed as an incomplete data problem. We first denote the observed and censored (missing) data as $\boldsymbol{Y} = (Y_{1:n}, Y_{2:n}, \ldots, Y_{m+k:n})$ and $\boldsymbol{Z} = (Z_1, Z_2, \ldots, Z_{n-m-k})$, respectively. Combine $\boldsymbol{Y}$ and $\boldsymbol{Z}$ to form the complete data $\boldsymbol{W}$. The E-step of the $(h+1)$-th EM-iteration requires the conditional expectations

$$\begin{aligned}
E\left(Z|\xi, \boldsymbol{\theta}^{(h)}\right) &= E_1\sigma^{(h)} + \mu^{(h)} \\
E\left(\exp\left(Z/\sigma\right)|\xi, \boldsymbol{\theta}^{(h)}\right) &= e^{\mu^{(h)}/\sigma^{(h)}}\left(e^{\xi}+1\right) \\
E\left(Z\exp\left(Z/\sigma\right)|\xi, \boldsymbol{\theta}^{(h)}\right) &= e^{\mu^{(h)}/\sigma^{(h)}}\left[\sigma^{(h)}E_2 + \mu^{(h)}\left(e^{\xi}+1\right)\right] \\
E\left(Z^2\exp\left(Z/\sigma\right)|\xi, \boldsymbol{\theta}^{(h)}\right) &= e^{\mu^{(h)}/\sigma^{(h)}}\left[\left(\sigma^{(h)}\right)^2 E_3\right. \\
&\quad \left. +2\mu^{(h)}\sigma^{(h)}E_2 - \left(\mu^{(h)}\right)^2\left(e^{\xi}+1\right)\right],
\end{aligned}$$

where $\boldsymbol{\theta}^{(h)} = (\mu^{(h)}, \sigma^{(h)})'$ is the current estimate of $\boldsymbol{\theta}$, $\xi = \frac{y_{m+k:n}-\mu^{(h)}}{\sigma^{(h)}}$ and

$$\begin{aligned}
E_1 &= E\left(\left.\frac{Z-\mu}{\sigma}\right|\xi, \boldsymbol{\theta}^{(h)}\right) \\
&= \psi(1)e^{e^{\xi}} + \sum_{p=0}^{\infty}\frac{e^{(p+1)\xi}\psi(p+2)}{\Gamma(p+2)} - [\xi+\psi(1)]\sum_{p=0}^{\infty}\frac{e^{(p+1)\xi}}{\Gamma(p+2)},
\end{aligned}$$

$$\begin{aligned}
E_2 &= E\left(\left.\frac{Z-\mu}{\sigma}\exp\left(\frac{Z-\mu}{\sigma}\right)\right|\xi, \boldsymbol{\theta}^{(h)}\right) \\
&= \psi(2)e^{e^{\xi}} + \sum_{p=0}^{\infty}\frac{e^{(p+2)\xi}\psi(p+3)}{\Gamma(p+3)} - [\xi+\psi(2)]\sum_{p=0}^{\infty}\frac{e^{(p+2)\xi}}{\Gamma(p+3)},
\end{aligned}$$

$$\begin{aligned}
E_3 &= E\left[\left.\left(\frac{Z-\mu}{\sigma}\right)^2\exp\left(\frac{Z-\mu}{\sigma}\right)\right|\xi, \boldsymbol{\theta}^{(h)}\right] \\
&= \left(\psi'(2)+\psi^2(2)\right)e^{e^{\xi}} \\
&\quad -\left(\xi^2+2\xi\psi(2)+\psi'(2)+\psi^2(2)\right)\sum_{p=0}^{\infty}\frac{e^{(p+2)\xi}}{\Gamma(p+3)} \\
&\quad +2\left(\xi+\psi(2)\right)\sum_{p=0}^{\infty}\frac{e^{(p+2)\xi}\psi(p+3)}{\Gamma(p+3)} \\
&\quad +\sum_{p=0}^{\infty}\frac{e^{(p+2)\xi}\left[\psi'(p+3)-\psi^2(p+3)\right]}{\Gamma(p+3)}.
\end{aligned}$$

The M-step in a EM-iteration is maximizing the likelihood based on complete sample over $\boldsymbol{\theta}$, with the missing values replaced by their conditional expectations. Thus, in the $(h+1)$-th EM-iteration, $\boldsymbol{\theta}^{(h+1)}$, is given by

$$\sigma^{(h+1)} = \frac{\sum_{j=1}^{m+k} y_{j:n} e^{y_{j:n}/\sigma^{(h+1)}} + (n-m-k)\, E\left(Z_1 \exp(Z_1/\sigma) \,|\xi, \boldsymbol{\theta}^{(h)}\right)}{\sum_{j=1}^{m+k} e^{y_{j:n}/\sigma^{(h+1)}} + (n-m-k)\, E\left(\exp(Z_1/\sigma) \,|\xi, \boldsymbol{\theta}^{(h)}\right)} - \frac{1}{n}\left[\sum_{j=1}^{m+k} y_{j:n} + (n-m-k)\, E\left(Z_1|\xi, \boldsymbol{\theta}^{(h)}\right)\right]$$

and

$$\mu^{(h+1)} = \sigma^{(h+1)} \log\left\{\frac{1}{n}\left[\sum_{j=1}^{m+k} e^{y_{j:n}/\sigma^{(h+1)}} + (n-m-k)\, E\left(\exp(Z_1/\sigma) \,|\xi, \boldsymbol{\theta}^{(h)}\right)\right]\right\}.$$

3.2. Asymptotic Variance-Covariance Matrix

Louis (1982) developed a procedure for extracting the observed information matrix when the EM-algorithm is used to obtain the MLEs based on incomplete data. The idea of the missing information principle can be expressed as (Louis, 1982 and Tanner, 1993)

Observed information = *Complete information* − *Missing information*.

In this subsection, we describe the use of the missing information for computing the conditional variance-covariance matrix of the MLEs under the MCS, given $K = k$. The observed information, complete information and missing information are denoted by $\boldsymbol{I_Y}(\boldsymbol{\theta})$, $\boldsymbol{I_W}(\boldsymbol{\theta})$ and $\boldsymbol{I_{Z|Y}}(\boldsymbol{\theta})$, respectively.

The complete information $\boldsymbol{I_W}(\boldsymbol{\theta})$ is given by

$$\boldsymbol{I_W}(\boldsymbol{\theta}) = -E\left(\frac{\partial^2 \lambda(\boldsymbol{\theta}; \boldsymbol{W})}{\partial \boldsymbol{\theta}^2}\right) = \frac{n}{\sigma^2}\begin{pmatrix} 1 & 1-\gamma \\ 1-\gamma & c^2 \end{pmatrix},$$

where $\gamma = 0.577215665\ldots$ is the Euler's constant and c^2 is $\pi^2/6 + (1-\gamma)^2$. The Fisher information matrix in one observation which is censored at the time of the m-th failure can be computed as

$$\boldsymbol{I}_{Z_j|Y}(\boldsymbol{\theta}) = -E\left(\frac{\partial^2 \log\left(f_{Zj}\left(Z_j | y_{m+k:n}, \boldsymbol{\theta}\right)\right)}{\partial \boldsymbol{\theta}^2}\right),\ j = 1, 2, \ldots, n-m-k,$$

where

$$\frac{\partial^2 \log\left(f_{Zj}\left(z_j|y_{m+k:n},\boldsymbol{\theta}\right)\right)}{\partial\mu^2} = \frac{1}{\sigma^2}\left(e^{\xi} - e^{\left(\frac{z_j-\mu}{\sigma}\right)}\right) \tag{3.3}$$

$$\frac{\partial^2 \log\left(f_{Zj}\left(z_j|y_{m+k:n},\boldsymbol{\theta}\right)\right)}{\partial\mu\partial\sigma} = \frac{1}{\sigma^2}\left\{1 + e^{\xi} + \xi e^{\xi} - e^{\left(\frac{z_j-\mu}{\sigma}\right)} - \left(\frac{z_j-\mu}{\sigma}\right)e^{\left(\frac{z_j-\mu}{\sigma}\right)}\right\} \tag{3.4}$$

$$\frac{\partial^2 \log\left(f_{Zj}\left(z_j|y_{m+k:n},\boldsymbol{\theta}\right)\right)}{\partial\sigma^2} = \frac{1}{\sigma^2}\left\{1 + \xi^2 e^{\xi} + 2\xi e^{\xi} - \left(\frac{z_j-\mu}{\sigma}\right)^2 e^{\left(\frac{z_j-\mu}{\sigma}\right)} -2\left(\frac{z_j-\mu}{\sigma}\right)e^{\left(\frac{z_j-\mu}{\sigma}\right)} + 2\left(\frac{z_j-\mu}{\sigma}\right)\right\}. \tag{3.5}$$

Then, the missing information matrix is

$$\boldsymbol{I}_{\boldsymbol{Z}|\boldsymbol{Y}}\left(\boldsymbol{\theta}\right) = (n-m-k)\,\boldsymbol{I}_{Z_1|\boldsymbol{Y}}\left(\boldsymbol{\theta}\right).$$

Therefore, the observed information matrix is presented by

$$\boldsymbol{I}_{\boldsymbol{Y}}\left(\boldsymbol{\theta}\right) = \boldsymbol{I}_{\boldsymbol{W}}\left(\boldsymbol{\theta}\right) - \boldsymbol{I}_{\boldsymbol{Z}|\boldsymbol{Y}}\left(\boldsymbol{\theta}\right). \tag{3.6}$$

The Fisher information matrix $\boldsymbol{I}_{\boldsymbol{Y}}\left(\boldsymbol{\theta}\right)$ can be obtained by the missing information principle. Hence, the asymptotic variance-covariance matrix of the MLEs of μ and σ can be obtained by inverting the observed information matrix $\boldsymbol{I}_{\boldsymbol{Y}}\left(\boldsymbol{\theta}\right)$.

Under some mild regularity conditions, the MLEs $\hat{\boldsymbol{\theta}} = (\hat{\mu}, \hat{\sigma})$ is approximately bivariate normal with mean $\boldsymbol{\theta} = (\mu, \sigma)$ and variance-covariance matrix $\boldsymbol{I}_{\boldsymbol{Y}}^{-1}\left(\boldsymbol{\theta}\right)$. In practice, $\boldsymbol{I}_{\boldsymbol{Y}}^{-1}\left(\boldsymbol{\theta}\right)$ is estimated by $\widehat{\boldsymbol{I}}_{Y}^{-1} = \boldsymbol{I}_{\boldsymbol{Y}}^{-1}\left(\widehat{\boldsymbol{\theta}}\right)$.

4. Monte Carlo Simulation Study

To evaluate the performance of the MLEs and the effect of the choice of progressive censoring scheme, a Monte Carlo simulation study is conducted. Without loss of generality, we consider the parameter setting $\mu = 0$ and $\sigma = 1$ with different values of n, m and different censoring schemes. Based on 1,000 simulations, the estimated biases, mean square errors (MSEs) and the average number of iterations of the EM-algorithm required to obtain the MLEs under MCS and the conventional progressive censoring scheme are presented in Table 1. Here, convergence is assumed when the absolute differences between the successive estimates are less than 10^{-5}. We also present the expected number of observed failures for the MCS (i.e., $E(K) + m$) in Table 1. As we expected, the biases and MSEs of the MLEs based on the MCS are smaller than or equal to those based on progressive censoring scheme with the same values of n and m. It is also observed that the average number of EM-iterations required to obtain MLEs based on the MCS are less than or equal to those based on progressive censoring scheme in all cases considered here. This is due to the fact that the number of observed failures is always m for the progressive censoring scheme while the number of observed failures in the MCS is a random variable with support greater than or equal to m. The MCS and progressive censoring scheme are equivalent when $\boldsymbol{R} = (0, \ldots, 0, n - m)$, which is also the Type-II censoring scheme.

In order to assess the accuracy of the approximation of the variances and covariance of the MLEs determined from the information matrix, the aforementioned simulation study is used. For the MCS, the simulated values of $Var(\widehat{\mu})$, $Var(\widehat{\sigma})$ and $Cov(\widehat{\mu}, \widehat{\sigma})$ as well as the estimated values by averaging the corresponding values obtained from the $\widehat{\boldsymbol{I}}_Y^{-1}$ are presented in Table 2. For the conventional progressive censoring schemes, the results are presented in Table 3 (Ng et al., 2002) to compare the simulation results of MCS. We can observe that the approximate values determined from $\widehat{\boldsymbol{I}}_Y^{-1}$ are quite close to simulated values for moderate values of m. Furthermore, it is noted that the approximation becomes more accurate as m increases.

Besides the biases and MSEs of the MLEs, we also compare the MCS and the progressive censoring scheme based on some commonly used optimal criteria in design of experiment. Since the parameter vector $\boldsymbol{\theta} = (\mu, \sigma)'$ is two-dimensional, optimality can be defined in terms of the following criteria (see, for example, Wu et al., 2008):

(1) *D-optimality*: Maximizing the determinant of the Fisher's information matrix. It is known that the determinant $|\boldsymbol{I}(\boldsymbol{\theta})|$ is proportional to the reciprocal of the volume of the asymptotic joint confidence region for $\boldsymbol{\theta}$ so that maximizing this determinant is equivalent to minimizing the volume of confidence region. Consequently, a larger value of the determinant of the Fisher's information matrix would correspond to higher joint precision of the estimators of $\boldsymbol{\theta}$.

(2) *A-optimality*: Maximizing the trace of the Fisher's information matrix. This optimal criterion is also known as trace criterion. It maximizes the sum of the diagonal entries of Fisher's information matrix. This means that the A-optimality criterion does not implement all available information the parameters.

(3) *E-optimality*: Maximizing the largest eigenvalue the Fisher's information matrix. This criterion maximizes the smallest non-zero eigenvalue of Fisher's information matrix. This also means that not all available information are used.

The average values of the objective functions for D-, A- and E-optimality criteria based on different MCSs and progressive censoring schemes are presented in Table 4. Since the expected number of observed failures for MCS is always $\geq m$, these values of the objective functions based on MCS are always better than those based on progressive censoring scheme with the same $\mathbf{R}$.

5. Illustrative Example

A real dataset from Lawless (1982) is used to illustrate the methodology developed in this paper. The dataset contains the failure times of $n = 15$ electrical insulating fluids that were subjected to a 32kV voltage stress. The dataset is presented in Table 5. If the lifetimes are assumed to be Weibull distributed, the MLEs of the Weibull parameters based on the complete dataset are $\widehat{\lambda} = 25.936$ and $\widehat{\beta} = 0.561$. Kolmogorov-Smirnov goodness-of-fit test statistic based on the complete dataset is 0.135 with p-value = 0.913, which indicates that Weibull distribution is a reasonable model for this dataset.

Suppose a progressive censoring plan with $m = 5$ and censoring scheme $\boldsymbol{R} = (10, 0, 0, 0, 0)$ are imposed in this life-testing experiment. Using this scheme, we generated the progressively censored sample is obtained by removing randomly selected 10 units from the test when the first failure is occurred at $x^{\boldsymbol{R}}_{1:5:15} = 0.27$ (i.e., at log-lifetime $y^{\boldsymbol{R}}_{1:5:15} = -1.3093$). The corresponding log-lifetimes and the 10 randomly selected units at the first observed failure (denoted by asterisk) are presented in Table 6. Hence, the progressively censored ordered log-lifetimes based on the progressive censoring scheme is (-1.3093, -0.2537, 1.0116, 4.4170, 4.4919).

Under the MCS, it is equivalent to use a Type-II censoring scheme where the first $m + K$ failures are observed. The p.m.f. of K is presented in Table 7.

Based on the random generation described above, we obtained $k = 8$ and the first 13 failures are observed.

The MLEs, the observed Fisher information matrices, the variance-covariance matrices and the values of the objective functions for D-, A- and E-optimality criteria based on complete sample ($n = 15$), progressively censored sample ($m = 5$) and the sample obtained from MCS ($m + k = 13$) are presented in Table 8. In this example, initial values for the EM-algorithm are chosen as $\mu_{(0)} = 1$ and $\sigma_{(0)} = 1$ in the two schemes and the number of EM-iterations required are 29 in both cases. The approximate 95% confidence intervals of μ and σ based on MCS are given by $(2.2798, 4.3466)$ and $(1.0281, 2.7323)$, respectively, while the corresponding 95% confidence intervals of μ and σ are obtained as $(1.8459, 4.7959)$ and $(0.7896, 2.5761)$ based on the progressive censoring scheme.

6. Conclusions

The progressive censoring scheme and the modified censoring scheme with the same censoring scheme $\boldsymbol{R}$ are terminated at the same time but the MCS collects more data on average. The relationship between the MCS and a randomized Type-II censoring scheme is discussed. Since more failures are observed based on the MCS, from the simulation results, it can be seen that the estimation of parameters based on MCS is more efficient than the progressive censoring scheme. Briefly, if the experimenter wants to estimate the distribution parameters efficiently and randomize the number of observed failures to be greater than m, then MCS scheme will be a suitable censoring scheme for the life test.

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References

1. Adamidis, K., Loukas, S. (1998). A lifetime distribution with decreasing failure rate, *Statistics and Probability Letters* 39: 35–42. DOI: 10.1016/S0167-7152(98)00012-1

2. Adamidis, K. (1999). An EM algorithm for estimating negative binomial parameters, *Australia and New Zealand Journal of Statistics* 41: 213–221. DOI: 10.1111/1467-842X.00075

3. Adamidis, K., Dimitrakopoulou, T., Loukas, S. (2005). On an extension of the exponential-geometric distribution, *Statistics and Probability Letters* 73: 259–269. DOI: 10.1016/j.spl.2005.03.013

4. Asgharzadeh, A. (2006). Point and interval estimation for a generalized logistic distribution under progressive Type-II censoring, *Communications in Statistics—Theory and Methods* 35: 1685–1702. DOI: 10.1080/03610920600683713

5. Bairamov, I., Parsi, S. (2011). On flexible progressive censoring, *Journal of Computational and Applied Mathematics* 235: 4537–4544. DOI: 10.1016/j.cam.2010.02.041

6. Balakrishnan, N. (2007). Progressive censoring methodology: an appraisal (with discussion), *TEST* 16: 211–289. DOI: 10.1007/s11749-007-0061-y

7. Balakrishnan, N., Aggarwala, R. (2000). *Progressive Censoring—Theory, Methods and Applications*. Birkhäuser, Boston.

8. Balakrishnan, N., Cramer, E. (2014). *The Art of Progressive Censoring: Applications to Reliability and Quality.* Birkhäuser, New York.

9. Balasooriya, U., Saw, S. L. C., Gadag, V. (2000). Progressively censored reliability sampling plans for the Weibull distribution, *Technometrics* 42: 160–167. DOI: 10.2307/1271448

10. Cohen, A. C. (1963). Progressively censored samples in the life testing, *Technometrics* 5: 327–339. DOI: 10.2307/1266337

11. Cramer, E., Iliopoulos, G. (2010). Adaptive Progressive Type-II Censoring, *TEST* 19: 342-358.

12. Gibbons, D. I., Vance, L. C. (1983). Estimators for the 2-parameter Weibull distribution with progressively censored samples, *IEEE Transactions on Reliability* 32: 95–99.

13. Karlis, D. (2003). An EM algorithm for multivariate Poisson distribution and related models, *Journal of Applied Statistics* 30: 63–77. DOI: 10.1080/0266476022000018510

14. Kundu, D., Joarder, A. (2006). Analysis of Type-II progressively hybrid censored data, *Computational Statistics and Data Analysis* 50: 2509–2258. DOI: 10.1016/j.csda.2005.05.002

15. Kuş, C. (2007). A new lifetime distribution, *Computational Statistics and Data Analysis* 51: 4497–4509. DOI: 10.1016/j.csda.2006.07.017

16. Kuş, C., Kaya, M. F. (2007). Estimation for the parameters of the Pareto distribution under progressive censoring, *Communications in Statistics– Theory and Methods* 36: 1359–1365. DOI: 10.1080/03610920601077089

17. Kuş, C., Wu, S.-J. (2008). Statistical inference based on progressive first failure-censoring samples from Gompertz distribution, In:*International Workshop on Applied Probability*, Université de Technologie de Compiégne, Compiégne, France, July 7–10.

18. Lawless, J. F. (1982). *Statistical Models and Methods for Lifetime Data.* Wiley, New York.

19. Lin, C. T., Ng, H. K. T., Chan, P. S. (2009). Statistical inference of Type-II progressively hybrid censored data with Weibull lifetimes, *Communications in Statistics–Theory and Methods* 38: 1710–1729. DOI: 10.1080/03610920902850069

20. Lio, Y. L., Tsai, T.-R. (2012). Estimation of $\delta = P(X < Y)$ for Burr XII distribution based on the progressively first failure-censored samples. *Journal of Applied Statistics* 39: 309–322. DOI: 10.1080/02664763.2011.586684

21. Louis, T. A. (1982). Finding the observed information matrix when using the EM algorithm, *Journal of Royal Statistical Society: Series B* 44: 226–233. DOI: 10.2307/2345828

22. Mann, N. R. (1971). Best linear invariant estimation for Weibull parameter under progressive censoring, *Technometrics* 13: 521–534. DOI: 10.2307/1267165

23. Morais, A. L., Barreto-Souza, W. (2011). A compound class of Weibull and power series distributions, *Computational Statistics and Data Analysis* 55: 1410–1425. DOI: 10.1016/j.csda.2010.09.030

24. Ng, H. K. T., Chan, P. S., Balakrishnan, N. (2002). Estimation of parameters from progressively censored data using EM algorithm, *Computational Statistics and Data Analysis* 39: 371–386.

25. Ng, H. K. T., Chan, P. S., Balakrishnan, N. (2004). Optimal progressive censoring plans for the Weibull distribution, *Technometrics* 46: 470–481. DOI: 10.1198/004017004000000482

26. Ng, H. K. T., Duan, F., Chan, P. S. (2014). On Conditional Moments of Progressively Censored Order Statistics with a Time Constraint, Submitted for publication.

27. Ng, H. K. T., Kundu, D., Chan, P. S. (2009). Statistical analysis of Exponential lifetimes under an adaptive Type-II progressive censoring scheme, *Naval Research Logistics* 56: 687–698. DOI: 10.1002/nav.20371

28. Saraçoglu, B., Kınacı, I., Kundu, D. (2012). On estimation of $R = P(Y < X)$ for exponential distribution under progressive Type-II censoring, *Journal of Statistical Computation and Simulation* 82: 729–744. DOI: 10.1080/00949655.2010.551772

29. Soliman, A. A. (2005). Estimation of parameters of life from progressively censored data using Burr-XII model, *IEEE Transactions on Reliability* 54: 34–42. DOI: 10.1109/TR.2004.842528

30. Soliman A. A., Abd-Ellah A. H., Abou-Elheggag, N. A., Abd-Elmougod G. A. (2012a). Estimation of the parameters of life for Gompertz distribution using progressive first-failure censored data, *Computational Statistics and Data Analysis* 56: 2471–2485. DOI: 10.1016/j.csda.2012.01.025

31. Soliman, A. A., Abd Ellah A. H., Abou-Elheggag, N. A., Modhesh, A. A. (2012b). Estimation of the coefficient of variation for non-normal model using progressive first-failure-censoring data, *Journal of Applied Statistics* 39: 2741–2758. DOI: 10.1080/02664763.2012.725466

32. Tahmasbi, R., Rezaei, S. (2008). A two-parameter lifetime distribution with decreasing failure rate, *Computational Statistics and Data Analysis* 52: 3889–3901. DOI: 10.1016/j.csda.2007.12.002

33. Tanner, M. A. (1993). *Tools for Statistical Inference. Observed Data and Data Augmentation Methods*, 2nd Edition. Springer, New York.

34. Viveros, R., Balakrishnan, N. (1994). Interval estimation of parameters of life from progressively censored data, *Technometrics* 36: 84–91. DOI: 10.2307/1269201

35. Wu, S.-J. (2002). Estimations of the parameters of the Weibull distribution with progressively censored data, *Journal of the Japan Statistical Society* 32: 155-163.

36. Wu, S.-J., Chen, D.-H., Chen, S.-T. (2006a). Bayesian inference for Rayleigh distribution under progressive censored sample, *Applied Stochastic Models in Business and Industry* 22: 269–279. DOI: 10.1002/asmb.615

37. Wu, S.-J., Lin, Y.-P., Chen, Y.-J. (2006b). Planning step-stress life test with progressively Type-I group-censored exponential data, *Statistica Neerlandica*, 60: 46–56. DOI: 10.1111/j.1467-9574.2006.00309.x

38. Wu, S.-J., Chen, Y.-J., Chang, C.-T. (2007). Statistical inference based on progressively censored samples with random removals from the Burr Type XII distribution, *Journal of Statistical Computation and Simulation* 77: 19–27. DOI:10.1080/10629360600569204

39. Wu, S.-J., Lin, Y.-P., Chen, S.-T. (2008). Optimal step-stress test under Type-I progressive group-censoring with random removals, *Journal of Statistical Planning and Inference* 138: 817–826. DOI: 10.1016/j.jspi.2007.02.004

40. Wu, S.-J., Kuş, C. (2009). On estimation based on progressive first-failure-censored sampling, *Computational Statistics and Data Analysis* 53: 3659–3670. DOI: 10.1016/j.csda.2009.03.010

41. Wu, S.-J., Huang, S.-R. (2012). Progressively first-failure censored reliability sampling plans with cost constraint, *Computational Statistics and Data Analysis* 56: 2018–2030. DOI: 10.1016/j.csda.2011.12.008

42. Yuen, H. K., Tse, S. K. (1996). Parameters estimation for Weibull distributed lifetime under progressive censoring with random removals, *Journal of Statistical Computation and Simulation* 55: 57–71. DOI: 10.1080/00949659608811749

Table 1. The estimated biases, MSEs, the average number of EM-iterations and the expected number of observed failures based on different MCSs and progressive censoring schemes

			Modified Censoring Scheme						Progressive Censoring Scheme				
			$\widehat{\mu}$		$\widehat{\sigma}$		$E(K)$	No.	$\widehat{\mu}$		$\widehat{\sigma}$		No.
n	m	R	Bias	MSE	Bias	MSE	$+m$	iter.	Bias	MSE	Bias	MSE	iter.
15	5	$(0,0,0,0,10)$	-0.2904	0.4541	-0.1901	0.1847	5.00	88.4	-0.2904	0.4541	-0.1901	0.1847	88.4
15	5	$(10,0,0,0,0)$	-0.0425	0.0909	-0.0539	0.0571	12.95	8.2	-0.1742	0.2676	-0.0815	0.0929	25.4
15	5	$(0,0,10,0,0)$	-0.0492	0.1011	-0.0671	0.0719	11.78	13.9	-0.2389	0.2916	-0.1307	0.0929	40.0
15	6	$(0,0,0,0,0,9)$	-0.2102	0.3234	-0.1405	0.1592	6.00	57.5	-0.2102	0.3234	-0.1405	0.1592	57.5
15	6	$(9,0,0,0,0,0)$	-0.0382	0.0815	-0.0546	0.0535	13.49	6.6	-0.1155	0.2041	-0.0819	0.0809	20.0
15	6	$(0,9,0,0,0,0)$	-0.0511	0.0845	-0.0671	0.0547	13.19	7.4	-0.1634	0.2268	-0.0960	0.0780	22.3
20	6	$(0,0,0,0,0,14)$	-0.2374	0.3938	-0.1384	0.1449	6.00	100.6	-0.2374	0.3938	-0.1384	0.1449	100.6
20	6	$(14,0,0,0,0,0)$	-0.0253	0.0583	-0.0529	0.0421	17.69	6.9	-0.1229	0.1903	-0.0741	0.0763	27.5
20	6	$(0,14,0,0,0,0)$	-0.0250	0.0662	-0.0529	0.0414	17.16	8.2	-0.1523	0.2237	-0.0882	0.0662	31.1
20	10	$(0,...,0,10)$	-0.1053	0.1348	-0.0770	0.0925	10.00	30.4	-0.1053	0.1348	-0.0770	0.0925	30.4
20	10	$(10,0,..,0)$	-0.0307	0.0585	-0.0345	0.0344	18.95	4.6	-0.0774	0.1138	-0.0468	0.0515	14.6
25	5	$(0,0,0,0,20)$	-0.3211	0.6352	-0.1676	0.1875	5.00	218.2	-0.3211	0.6352	-0.1676	0.1875	218.2
25	5	$(20,0,0,0,0)$	-0.0320	0.0564	-0.0366	0.0363	20.98	9.1	-0.1882	0.2713	-0.0641	0.0744	43.6
25	5	$(0,20,0,0,0)$	-0.0294	0.0616	-0.0410	0.0389	20.28	10.8	-0.2092	0.2721	-0.0759	0.0739	51.8
25	15	$(0,...,0,10)$	-0.0691	0.0794	-0.0535	0.0535	15.00	18.4	-0.0691	0.0794	-0.0535	0.0535	18.4
25	15	$(10,0,...,0)$	-0.0144	0.0460	-0.0323	0.0264	24.38	3.5	-0.0290	0.0727	-0.0348	0.0380	11.3
30	3	$(0,0,27)$	-0.4720	1.3382	-0.1557	0.2186	3.00	689.6	-0.4720	1.3382	-0.1557	0.2186	689.6
30	3	$(27,0,0)$	-0.0442	0.0774	-0.0405	0.0450	21.23	25.3	-0.3534	0.5805	-0.0940	0.1062	103.6
30	4	$(0,0,0,26)$	-0.4250	1.2194	-0.1656	0.2512	4.00	444.3	-0.4250	1.2194	-0.1656	0.2512	444.3
30	4	$(26,0,0,0)$	-0.0238	0.0539	-0.0353	0.0345	23.70	12.3	-0.2223	0.3504	-0.0726	0.0889	68.1
50	20	$(0,...,0,30)$	-0.0713	0.0786	-0.0491	0.0452	20.00	45.9	-0.0713	0.0786	-0.0491	0.0452	45.9
50	20	$(30,0,...,0)$	-0.0059	0.0219	-0.0153	0.0125	48.48	4.1	-0.0350	0.0541	-0.0192	0.0252	18.0
50	25	$(0,...,0,25)$	-0.0449	0.0541	-0.0260	0.0359	25.00	26.9	-0.0449	0.0541	-0.0260	0.0359	26.9
50	25	$(25,0,...,0)$	-0.0017	0.0228	-0.0197	0.0128	49.02	3.6	-0.0153	0.0445	-0.0240	0.0205	13.9

Table 2. Estimated variances and covariance of the MLEs under MCSs based on simulation and averaging the observed information matrices

			Simulated			From observed information		
n	m	$\boldsymbol{R}$	Var($\widehat{\mu}$)	Var($\widehat{\sigma}$)	Cov($\widehat{\mu},\widehat{\sigma}$)	Var($\widehat{\mu}$)	Var($\widehat{\sigma}$)	Cov($\widehat{\mu},\widehat{\sigma}$)
15	5	$(0,0,0,0,10)$	0.3698	0.1486	0.1503	0.2999	0.1407	0.1398
15	5	$(10,0,0,0,0)$	0.0891	0.0542	-0.0066	0.0811	0.0529	-0.0047
15	5	$(0,0,10,0,0)$	0.0986	0.0674	0.0061	0.0933	0.0630	0.0067
15	6	$(0,0,0,0,0,9)$	0.2792	0.1394	0.1155	0.2254	0.1261	0.0998
15	6	$(9,0,0,0,0,0)$	0.0800	0.0505	-0.0093	0.0752	0.0480	-0.0094
15	6	$(0,9,0,0,0,0)$	0.0819	0.0502	-0.0074	0.0755	0.0490	-0.0071
20	6	$(0,0,0,0,0,14)$	0.3374	0.1258	0.1380	0.3053	0.1295	0.1442
20	6	$(14,0,0,0,0,0)$	0.0576	0.0393	-0.0040	0.0571	0.0371	-0.0058
20	6	$(0,14,0,0,0,0)$	0.0656	0.0387	-0.0029	0.0598	0.0395	-0.0034
20	10	$(0,...,0,10)$	0.1238	0.0865	0.0466	0.1167	0.0789	0.0425
20	10	$(10,0,..,0)$	0.0576	0.0332	-0.0112	0.0549	0.0332	-0.0102
25	5	$(0,0,0,0,20)$	0.5321	0.1594	0.2327	0.5444	0.1562	0.2417
25	5	$(20,0,0,0,0)$	0.0553	0.0349	-0.0002	0.0515	0.0342	-0.0009
25	5	$(0,20,0,0,0)$	0.0607	0.0372	0.0049	0.0553	0.0366	0.0022
25	15	$(0,...,0,10)$	0.0746	0.0506	0.0186	0.0683	0.0512	0.0164
25	15	$(10,0,...,0)$	0.0458	0.0253	-0.0091	0.0430	0.0248	-0.0092
30	3	$(0,0,27)$	1.4779	0.2132	0.4824	1.2616	0.2032	0.4607
30	3	$(27,0,0)$	0.0755	0.0433	0.0191	0.0756	0.0413	0.0183
30	4	$(0,0,0,26)$	0.9899	0.1804	0.3576	0.8728	0.1760	0.3471
30	4	$(26,0,0,0)$	0.0533	0.0332	0.0055	0.0510	0.0331	0.0049
50	20	$(0,...,0,30)$	0.0735	0.0428	0.0334	0.0741	0.0418	0.0334
50	20	$(30,0,...,0)$	0.0218	0.0123	-0.0041	0.0220	0.0129	-0.0046
50	25	$(0,...,0,25)$	0.0521	0.0352	0.0179	0.0493	0.0336	0.0183
50	25	$(25,0,...,0)$	0.0228	0.0124	-0.0044	0.0217	0.0125	-0.0047

Table 3. Estimated variances and covariance of the MLEs under progressive censoring schemes based on simulation and averaging the observed information matrices

			Simulated			From observed information		
n	m	$\boldsymbol{R}$	Var($\widehat{\mu}$)	Var($\widehat{\sigma}$)	Cov($\widehat{\mu},\widehat{\sigma}$)	Var($\widehat{\mu}$)	Var($\widehat{\sigma}$)	Cov($\widehat{\mu},\widehat{\sigma}$)
15	5	$(0,0,0,0,10)$	0.3676	0.1522	0.1549	0.3167	0.1507	0.1510
15	5	$(10,0,0,0,0)$	0.2194	0.0855	-0.0004	0.1874	0.0816	-0.0111
15	5	$(0,0,10,0,0)$	0.2558	0.0874	0.0550	0.1906	0.0750	0.0378
15	6	$(0,0,0,0,0,9)$	0.2544	0.1182	0.0984	0.2147	0.1220	0.0974
15	6	$(9,0,0,0,0,0)$	0.1751	0.0707	-0.0046	0.1543	0.0713	-0.0140
15	6	$(0,9,0,0,0,0)$	0.1785	0.0660	0,0119	0.1482	0.0620	0.0020
20	6	$(0,0,0,0,0,14)$	0.3320	0.1211	0.1390	0.2936	0.1256	0.1404
20	6	$(14,0,0,0,0,0)$	0.1754	0.0652	0,0001	0.1559	0.0661	-0.0092
20	6	$(0,14,0,0,0,0)$	0.1835	0.0593	0,0190	0.1518	0.0554	0.0084
20	10	$(0,...,0,10)$	0.1302	0.0817	0.0477	0.1183	0.0811	0.0440
20	10	$(10,0,..,0)$	0.1051	0.0508	-0.0107	0.1020	0.0502	-0.0146
25	5	$(0,0,0,0,20)$	0.5928	0.1567	0.2450	0.5421	0.1563	0.2428
25	5	$(20,0,0,0,0)$	0.2232	0.0739	0.0084	0.1922	0.0707	-0.0020
25	5	$(0,20,0,0,0)$	0.2535	0.0657	0.0418	0.1924	0.0561	0.0249
25	15	$(0,...,0,10)$	0.0757	0.0530	0.0194	0.0679	0.0516	0.0167
25	15	$(10,0,...,0)$	0.0713	0.0359	-0.0095	0.0679	0.0349	-0.0119
30	3	$(0,0,27)$	1.4423	0.1994	0.4612	1.2690	0.2047	0.4656
30	3	$(27,0,0)$	0.4693	0.0929	0.0730	0.3084	0.0772	0.0285
30	4	$(0,0,0,26)$	0.9209	0.1819	0.3450	0.9019	0.1823	0.3606
30	4	$(26,0,0,0)$	0.2876	0.0789	0.0177	0.2393	0.0720	0.0110
50	20	$(0,...,0,30)$	0.0769	0.0444	0.0354	0.0763	0.0432	0.0346
50	20	$(30,0,...,0)$	0.0511	0.0254	-0.0079	0.0521	0.0250	-0.0079
50	25	$(0,...,0,25)$	0.0482	0.0330	0.0187	0.0485	0.0332	0.0181
50	25	$(25,0,...,0)$	0.0397	0.0210	-0.0064	0.0417	0.0211	-0.0074

Table 4. Average of the value of objective functions under different optimality criteria for progressive censoring schemes and MCSs

			PC			MPC		
n	m	R	D Opt.	E Opt.	A Opt.	D Opt.	E Opt.	A Opt.
15	5	$(0,0,0,0,10)$	319.3455	38.7014	5.1989	319.3455	38.7014	5.1989
15	5	$(10,0,0,0,0)$	169.3440	24.3372	6.7084	601.0517	46.1881	15.7142
15	5	$(0,0,10,0,0)$	164.1782	25.2080	5.9831	569.2904	44.2240	15.5566
15	6	$(0,0,0,0,0,9)$	361.2027	36.3757	6.6444	361.2027	36.3757	6.6444
15	6	$(9,0,0,0,0,0)$	194.2819	26.5514	7.5731	612.3198	47.2612	15.5816
15	6	$(0,9,0,0,0,0)$	204.5630	28.0045	7.8681	613.9599	47.5048	15.8701
20	6	$(0,0,0,0,0,14)$	392.3525	46.8872	5.2405	392.3525	46.8872	5.2405
20	6	$(14,0,0,0,0,0)$	207.2707	28.6468	7.7592	871.5973	57.7428	19.7628
20	6	$(0,14,0,0,0,0)$	209.1496	29.5509	7.4431	857.2490	56.8853	19.5341
20	10	$(0,...,0,10)$	495.0468	41.2053	10.8037	495.0468	41.2053	10.8037
20	10	$(10,0,..,0)$	404.6000	39.8510	11.6485	1013.0154	63.7111	20.0633
25	5	$(0,0,0,0,20)$	386.3470	59.7886	3.4387	386.3470	59.7886	3.4387
25	5	$(20,0,0,0,0)$	174.5733	26.8973	6.5886	1087.5170	64.4750	22.8599
25	5	$(0,20,0,0,0)$	202.1149	30.7781	6.2213	1052.6745	62.9574	22.6072
25	15	$(0,...,0,10)$	599.1035	46.0092	15.8704	599.1035	46.0092	15.8704
25	15	$(10,0,...,0)$	659.1227	52.8930	15.5362	1434.1949	78.3677	23.8477
30	3	$(0,0,27)$	456.0452	96.9455	2.0277	456.0452	96.9455	2.0277
30	3	$(27,0,0)$	124.6719	24.0509	4.2643	1187.8925	66.8439	23.4095
30	4	$(0,0,0,26)$	443.5982	84.4939	2.6257	443.5982	84.4939	2.6257
30	4	$(26,0,0,0)$	145.0492	25.2754	5.3242	1279.0570	70.2054	24.8659
50	20	$(0,...,0,30)$	806.4611	67.8134	12.1865	806.4611	67.8134	12.1865
50	20	$(30,0,...,0)$	969.1521	65.9189	18.9523	3596.1957	127.6176	39.9496
50	25	$(0,...,0,25)$	1140.5915	71.8385	18.6985	1140.5915	71.8385	18.6985
50	25	$(25,0,...,0)$	1283.2644	76.5887	22.2807	3725.8131	131.0560	39.9545

Table 5. Complete ordered lifetime data from Lawless (1982)

0.27	0.40	0.69	0.79	2.75	3.91	9.88	13.95	15.93	27.80
53.24	82.85	89.29	100.58	215.10					

Table 6. Complete ordered log-lifetime data presented by Lawless (1982)

-1.3093	*-0.9163	*-0.3711	-0.2357	1.0116	*1.3635	*2.2905	*2.6355
*2.7682	*3.3250	*3.9748	4.4170	4.4919	*4.6110	*5.3711	

Table 7. Probability mass function of K in the illustrative example (the number of observed failure is $m + K$) based on the MCS

k	0	1	2	3	4	5
$\Pr(K = k)$	0.285714	0.219780	0.164835	0.119880	0.083916	0.055944
k	6	7	8	9	10	
$\Pr(K = k)$	0.034965	0.019980	0.009990	0.003996	0.000999	

Table 8. Results for the illustrative example in Section 5

	Complete Sample	MPC Scheme	PC Scheme
$\begin{pmatrix} \widehat{\mu} \\ \widehat{\sigma} \end{pmatrix}$	$\begin{pmatrix} 3.2556 \\ 1.7812 \end{pmatrix}$	$\begin{pmatrix} 3.3132 \\ 1.8802 \end{pmatrix}$	$\begin{pmatrix} 3.3209 \\ 1.6828 \end{pmatrix}$
$\widehat{I}_Y\left(\widehat{\theta}\right)$	$\begin{pmatrix} 4.728 & 1.999 \\ 1.999 & 8.622 \end{pmatrix}$	$\begin{pmatrix} 3.677 & 0.659 \\ 0.659 & 5.410 \end{pmatrix}$	$\begin{pmatrix} 1.766 & 0.004 \\ 0.004 & 4.815 \end{pmatrix}$
$\widehat{I}_Y^{-1}\left(\widehat{\theta}\right)$	$\begin{pmatrix} 0.235 & -0.054 \\ -0.054 & 0.129 \end{pmatrix}$	$\begin{pmatrix} 0.278 & -0.034 \\ -0.034 & 0.189 \end{pmatrix}$	$\begin{pmatrix} 0.566 & -0.001 \\ -0.001 & 0.208 \end{pmatrix}$
D-optimal	36.765	19.462	8.502
A-optimal	13.349	9.088	6.581
E-optimal	3.884	3.456	1.766